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BRIEF COMMUNICATIONS

On Series with Respect to the Rademacher System in Rearrangement Invariant Spaces "Close" to L_∞

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Let

$$r_k(t) = \operatorname{sign}\sin 2^{k-1}\pi t \qquad (k = 1, 2, \dots)$$

be the system of Rademacher functions on $[0,1]$.

For $a = (a_k)_{k=1}^\infty \in l_2$ we define the linear operator

$$Ta(t) = \sum_{k=1}^{\infty} a_k r_k(t) \qquad (t \in [0,1]). \tag{1}$$

If $\|a\|_p = (\sum_{k=1}^\infty |a_k|^p)^{1/p}$ as usual, then it follows from the Khinchin inequality that

$$\|Ta\|_{L_p[0,1]} \asymp \|a\|_2 \tag{2}$$

for $1 \le p < \infty$ (this means the existence of two-sided estimates with constants depending on p only). Moreover, we can readily verify the relation

$$\|Ta\|_{L_\infty[0,1]} = \|a\|_1. \tag{3}$$

More detailed information on the behavior of series with respect to the Rademacher system can be obtained by treating these series in the framework of general rearrangement invariant spaces.

Recall that a Banach space X of measurable functions $x = x(t)$ on $[0,1]$ is said to be rearrangement invariant (r.i.s.) if the relations $x^*(t) \le y^*(t)$ $(t \in [0,1])$ and $y \in X$ imply the relations $x \in X$ and $\|x\| \le \|y\|$ (where $z^*(t)$ is the nonincreasing rearrangement of a function $|z(t)|$ [1, p. 83]). For any r.i.s. X on $[0,1]$ we have $L_\infty \subset X \subset L_1$ [1, p. 124].

An Orlicz space is an important example of an r.i.s. Let $N(t)$ be an increasing convex function on $[0,\infty)$ such that $N(0) = 0$ and $\lim_{u\to 0} N(u)/u = \lim_{u\to\infty} u/N(u) = 0$. The Orlicz space L_N consists of all measurable functions $x = x(t)$ on $[0,1]$ such that

$$\|x\|_N = \inf\left\{u > 0 : \int_0^1 N\left(\frac{|x(t)|}{u}\right) dt \le 1\right\} < \infty.$$

In particular, if $N(t) = t^p$ $(1 \le p < \infty)$, then $L_N = L_p$. In the problems discussed below, a special role is played by the Orlicz space L_M, where $M(t) = \exp(t^2) - 1$, or, more precisely, the closure of L_∞ in the space L_M, which is denoted by G.

As was proved in [2], the norm of an r.i.s. X is equivalent to the norm of l_2 on the linear span of the Rademacher functions (i.e., a relation similar to (2) holds for X) if and only if $X \supset G$.

In this note we present results on the behavior of series with respect to the Rademacher system in an r.i.s. X that is intermediate between L_∞ and G. The main role here is played by notions and methods of the interpolation theory of linear operators.

For a Banach pair (X_0, X_1), $x \in X_0 + X_1$, and $t > 0$, we introduce the Peetre $\mathscr{K}$-functional as follows:

$$\mathscr{K}(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1,\ x_i \in X_i\}.$$

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Let Y_0 be a subspace of X_0 and let Y_1 be a subspace of X_1. A pair (Y_0, Y_1) is called a $\mathscr{K}$-subpair of the pair (X_0, X_1) if

$$\mathscr{K}(t, y; Y_0, Y_1) \asymp \mathscr{K}(t, y; X_0, X_1)$$

with constants that do not depend on $y \in Y_0 + Y_1$ and $t > 0$.

In particular, if $Y_i = P(X_i)$ $(i = 0, 1)$, where P is a bounded linear projection in X_0 and X_1, then (Y_0, Y_1) is a $\mathscr{K}$-subpair of (X_0, X_1) [3, p. 136 of the Russian translation]. At the same time, there are many examples of subpairs that are not $\mathscr{K}$-subpairs (see [3, p. 589 of the Russian translation; 4] and Remark 1 of the present note).

Consider the case in which $X_0 = L_\infty$, $X_1 = G$, $Y_0 = T(l_1)$, and $Y_1 = T(l_2)$, where T is the operator given by (1). It follows from relation (3) and from the above statement of [2] that

$$\mathscr{K}(t, Ta; T(l_1), T(l_2)) \asymp \mathscr{K}(t, a; l_1, l_2). \tag{4}$$

In spite of the fact that $T(l_1)$ is not complemented in L_∞ [5], the following statement holds.

Theorem 1. *The pair $(T(l_1), T(l_2))$ is a $\mathscr{K}$-subpair of pair (L_∞, G). In other words, by* (4)

$$\mathscr{K}(t, Ta; L_\infty, G) \asymp \mathscr{K}(t, a; l_1, l_2)$$

with constants that do not depend on $a = (a_k)_{k=1}^\infty \in l_2$ and $t > 0$.

In the proof of Theorem 1, the result of [6] on the distribution of Rademacher sums is used.

Remark 1. The assertion of Theorem 1 fails if the space G is replaced by L_p for any $p < \infty$.

Every r.i.s. X naturally generates the coordinate sequence space of the coefficients of series with respect to the Rademacher system. Theorem 1 allows one to find this coordinate space for the case in which X is an interpolation space between L_∞ and G (that is, $L_\infty \subset X \subset G$ and each linear operator bounded in L_∞ and G is bounded in X).

Let E be a Banach ideal space of two-sided sequences such that $(\min(1, 2^k))_{k=-\infty}^\infty \in E$. If (X_0, X_1) is a Banach pair, then the space $(X_0, X_1)_E^{\mathscr{K}}$ corresponding to the real $\mathscr{K}$-method of interpolation consists of all $x \in X_0 + X_1$ such that

$$\|x\| = \|(\mathscr{K}(2^k, x; X_0, X_1))_k\|_E < \infty.$$

In the special case $E = l_p(2^{-k\theta})$, $0 < \theta < 1$, $1 \le p \le \infty$, we obtain the classical spaces $(X_0, X_1)_{\theta,p}$ (for the detailed exposition of their properties, see [7]).

If an r.i.s. X is an interpolation space between L_∞ and G, then it is described by the real $\mathscr{K}$-method [8]. This means that

$$X = (L_\infty, G)_E^{\mathscr{K}} \tag{5}$$

for some parameter E. Therefore, Theorem 1 implies the following assertion.

Theorem 2. *Let r.i.s. X be an interpolation space between L_∞ and G and let relation* (5) *hold. In this case, for the sequence space $F = (l_1, l_2)_E^{\mathscr{K}}$ we have*

$$\left\| \sum_{k=1}^\infty a_k r_k \right\|_X \asymp \|(a_k)\|_F.$$

This result shows that the correspondence established in Theorem 2 is one-to-one.

Theorem 3. *Let r.i.s.'s X_0 and X_1 be two interpolation spaces between L_∞ and G. If*

$$\left\| \sum_{k=1}^\infty a_k r_k \right\|_{X_0} \asymp \left\| \sum_{k=1}^\infty a_k r_k \right\|_{X_1},$$

then $X_0 = X_1$, and the norms in X_0 and X_1 are equivalent.

The above results allow one to find the sequence spaces of the coefficients of series with respect to the Rademacher system for concrete r.i.s.'s.

In the examples below, $\varphi \geq 0$ is an increasing concave function.

Example 1. The Lorentz r.i.s. $\Lambda_p(\varphi)$ $(1 \leq p < \infty)$ consists of all measurable functions $x = x(s)$ such that

$$\|x\|_{\varphi,p} = \left\{ \int_0^1 (x^*(s))^p \, d\varphi(s) \right\}^{1/p} < \infty .$$

In particular, if $\varphi(s) = \log_2^{1-p} 2/s$ and $1 < p < 2$, then

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{\varphi,p} \asymp \|(a_k)\|_p .$$

Example 2. The Marcinkiewicz r.i.s. $M(\varphi)$ consists of all measurable functions $x = x(s)$ such that

$$\|x\|_{M(\varphi)} = \sup \left\{ \frac{1}{\varphi(t)} \int_0^t x^*(s) \, ds : 0 < t \leq 1 \right\} < \infty .$$

In particular, if $\varphi(t) = t \log_2 \log_2 16/t$, then

$$\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{M(\varphi)} \asymp \|(a_k)\|_{l_1(\ln)} = \sup \left\{ \frac{1}{\log_2(2k)} \sum_{i=1}^{k} a_i^* : k = 1, 2, \dots \right\},$$

where (a_i^*) is the nonincreasing rearrangement of a sequence $(|a_k|)_{k=1}^{\infty}$.

Remark 2. Theorems 2 and 3 strengthen results of [2], where similar statements were obtained for sequence spaces F satisfying more restrictive conditions. For instance, we can readily show that the norm of the dilation operator

$$\sigma_n a = (\underbrace{a_1, \dots, a_1}_{n}, \underbrace{a_2, \dots, a_2}_{n}, \dots)$$

in the space $l_1(\ln)$ (see Example 2) is equal to n. Therefore, condition (11) of [2] fails for this space, and the theorems obtained in [2] cannot be applied to this operator.

Remark 3. Theorems 1–3 can readily be extended to lacunary trigonometric series by means of results in [9].

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