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INTERPOLATION OF POSITIVE POLYLINEAR OPERATORS IN CALDERÓN–LOZANOVSKIĬ SPACES

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1. Preliminaries and statement of the results. Let Φ be the class of functions $\varphi = \varphi(u,v)$ defined for $u \geq 0$, $v \geq 0$, positive homogeneous of degree one, and increasing in each argument. Given a pair $\{E_0, E_1\}$ of Banach lattices (BL) of measurable functions on a space with a σ-finite measure μ, denote by $\varphi(E_0, E_1)$ the space of all measurable functions $x = x(t)$ for which

$$|x(t)| \leq \lambda\varphi(x_0(t), x_1(t)), \tag{1}$$

where $\lambda > 0$, $x_i \geq 0$, $x_i \in E_i$, and $\|x_i\|_{E_i} \leq 1$ $(i = 0, 1)$. The norm in $\varphi(E_0, E_1)$ is defined to be $\inf \lambda$ calculated over all λ satisfying (1).

This construction first appeared in A. P. Calderón's article [1]. Thereafter its various aspects were studied by a number of mathematicians. In particular, G. Ya. Lozanovskiĭ demonstrated that in general the space $\varphi(E_0, E_1)$ is not an interpolation space between E_0 and E_1 [2]. This means that not every linear operator bounded in E_0 and E_1 is bounded in $\varphi(E_0, E_1)$. At the same time, if a linear operator is positive then the following interpolation theorem is valid: boundedness of such operator in E_0 and E_1 implies its boundedness in $\varphi(E_0, E_1)$ [3].

In the present article, we prove a similar assertion for polylinear operators. Stating the results, we need some definitions and notations.

Given a nonnegative function f on $(0, \infty)$, define its dilation function $\mathcal{M}_f$ by the equality

$$\mathcal{M}_f(t) = \sup_{s>0} \frac{f(st)}{f(s)} \quad (t > 0).$$

Since $\mathcal{M}_f$ is polymultiplicative, there are numbers

$$\gamma_f = \lim_{t\to 0} \frac{\log \mathcal{M}_f(t)}{\log t}, \quad \delta_f = \lim_{t\to\infty} \frac{\log \mathcal{M}_f(t)}{\log t}$$

called the *lower* and *upper dilation exponents* of f.

Given $\varphi \in \Phi$, put $\rho(t) = \varphi(1, t)$. It is easy to demonstrate that $0 \leq \gamma_\rho \leq \delta_\rho \leq 1$. Henceforth we suppose that the function ρ satisfies the following two conditions:

$$\rho(u)\rho(v) \leq K\rho(uv), \tag{*}$$

with K independent of $u > 0$ and $v > 0$, and

$$0 < \gamma_\rho \leq \delta_\rho < 1. \tag{**}$$

Suppose that A is a polylinear operator acting boundedly from the product $\prod_{i=1}^n X_i$ of Bls X_i into a BL Y. The operator A is called *positive* if $x_1 \geq 0, \dots, x_n \geq 0$ implies that $A(x_1, \dots, x_n) \geq 0$.

If X is a BL of measurable functions then the dual space X' comprises all measurable functions $y(t)$ such that

$$\|y\|_{E'} = \sup\left\{ \int_T x(t)y(t)\,d\mu : \|x\|_E \leq 1 \right\} < \infty.$$

The second dual is denoted by X'' as usual.

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Theorem 1. *Suppose that A is a bounded positive polylinear operator from $\prod_{i=1}^{n} X_0^i$ into Y_0 and from $\prod_{i=1}^{n} X_1^i$ into Y_1. If a function $\varphi \in \Phi$ satisfies the conditions $(*)$ and $(**)$ then A acts boundedly from $\prod_{i=1}^{n} \varphi(X_0^i, X_1^i)$ into $\varphi(Y_0'', Y_1'') \cap (Y_0 + Y_1)$.*

Theorem 1 is a consequence of, first, the interpolation theorem for concrete weighted spaces of two-sided numeric sequences (Theorem 2) and, second, the possibility of describing the Calderón–Lozanovskiĭ method in terms of the orbits of positive operators (Theorem 3).

Suppose that $w = \{w_k\}_{k=-\infty}^{\infty}$ is a sequence of nonnegative numbers and E is a BL of numeric sequences. Denote by $E(w) = E(w_k)$ the BL of sequences $x = (x_k)_{k=-\infty}^{\infty}$ with the finite norm $\|x\|_{E(w)} = \|(w_k x_k)\|_E$. Henceforth the notations l_p and c_0 are understood in a standard way and $\varphi^*(u,v) = 1/\varphi(1/u, 1/v)$.

Theorem 2. *Suppose that the function $\rho(t) = \varphi(1,t)$, $\varphi \in \Phi$, satisfies the conditions $(*)$ and $(**)$. If a positive polylinear operator V acts boundedly from $\prod_{i=1}^{n} c_0$ into l_1 and from $\prod_{i=1}^{n} c_0(2^{-k})$ into $l_1(2^{-k})$ then it is a bounded operator from $\prod_{i=1}^{n} l_\infty(\varphi^*(1, 2^{-k}))$ into $l_1(\varphi^*(1, 2^{-k}))$.*

Recall that a linear operator A is said to be a *bounded operator from a Banach pair $\vec{X} = \{X_0, X_1\}$ into a pair $\vec{Y} = \{Y_0, Y_1\}$* if A is a bounded operator from X_0 into Y_0 and from X_1 into Y_1. Moreover,

$$\|A\|_{\vec{X} \to \vec{Y}} = \max_{i=0,1}\{\|A\|_{X_i \to Y_i}\}.$$

We now define the orbit and the co-orbit of a space with respect to the class $L(\vec{X}, \vec{Y})$ of all bounded linear operators from the pair $\vec{X} = \{X_0, X_1\}$ into the pair $\vec{Y} = \{Y_0, Y_1\}$.

Suppose that $\vec{E} = \{E_0, E_1\}$ is a Banach pair and E is an intermediate Banach space for $\vec{E}$; i.e., $E_0 \cap E_1 \subset E \subset E_0 + E_1$. Then the *orbit* of E in the pair $\vec{Y} = \{Y_0, Y_1\}$ is defined to be the space $\mathrm{Orb}_{\vec{E}}(E, \vec{Y})$ of all $y \in Y_0 + Y_1$ representable as

$$y = \sum_{j=1}^{\infty} T_j x_j, \tag{2}$$

where the series converges in $Y_0 + Y_1$, $T_j \in L(\vec{E}, \vec{Y})$, $x_j \in E$, and

$$\sum_{j=1}^{\infty} \|T_j\|_{\vec{E} \to \vec{Y}} \|x_j\|_E < \infty. \tag{3}$$

The norm $\|y\|_{\mathrm{Orb}}$ is defined as the greatest lower bound of the sums in (3) over all representations of the form (2).

The space of all $y \in Y_0 + Y_1$ with

$$\|y\|_{\mathrm{Corb}} = \sup\{\|Ty\|_E : \|T\|_{\vec{Y} \to \vec{E}} \le 1\} < \infty$$

is called the *co-orbit* of E in the pair $\vec{Y}$ and denoted by $\mathrm{Corb}_{\vec{E}}(E, \vec{Y})$.

Extremely important for interpolation theory is the possibility of describing a series of most important classes of interpolation functors in terms of orbits and co-orbits (see, for instance, [4]). Considering only pairs of Bls in the above definitions and confining the exposition to positive operators, we obtain the definitions of $\mathrm{Orb}^+_{\vec{E}}(E, \vec{Y})$ and $\mathrm{Corb}^+_{\vec{E}}(E, \vec{Y})$. It is clear that

$$\mathrm{Orb}^+_{\vec{E}}(E, \vec{Y}) \subset \mathrm{Orb}_{\vec{E}}(E, \vec{Y}), \mathrm{Corb}^+_{\vec{E}}(E, \vec{Y}) \supset \mathrm{Corb}_{\vec{E}}(E, \vec{Y}).$$

In [5], the following result was obtained on describing the positive operators of the Calderón–Lozanovskiĭ method in terms of orbits:

Theorem 3. *If $\varphi \in \Phi$ and $\vec{l}_p = \{l_p, l_p(2^{-k})\}$ then the following relations hold for an arbitrary pair $\vec{Y} = \{Y_0, Y_1\}$ of* Bls:

$$\varphi(\vec{Y}) = \mathrm{Orb}^+_{\vec{l}_\infty}(l_\infty(\varphi^*(1,2^{-k})), \vec{Y}), \tag{4}$$

$$\varphi(\vec{Y}'') \cap (Y_0 + Y_1) \supset \mathrm{Corb}^+_{\vec{l}_1}(l_1(\varphi^*(1,2^{-k})), \vec{Y}), \tag{5}$$

where $\vec{Y}'' = \{Y_0'', Y_1''\}$.

REMARK 1. We may also obtain the embedding (5) by slightly modifying the proof of Lemma 8.5.1 of [4].

REMARK 2. For linear (not necessarily positive) operators, theorems similar to Theorems 1 and 2 were first proven in [6] for an arbitrary function $\varphi \in \Phi$. It is easy to demonstrate that the condition $(*)$ is necessary in the assertions of Theorems 1 and 2 even in the bilinear case. At the same time, the question remains open whether the exponents of the dilation function $\rho(t) = \varphi(1,t)$ (the condition $(**)$) must be nontrivial.

REMARK 3. In the power case $\varphi(u,v) = u^{1-s}v^s$ $(0 < s < 1)$, we may obtain Theorem 1 and 2 by using the connection between the Calderón–Lozanovskiĭ method and the complex interpolation method together with the polylinear interpolation theorem for the latter (see [1] or [7, pp. 125–128]).

REMARK 4. In the bilinear case Theorem 1 and 2 were announced in the Proceedings of the International Conference dedicated to the 90th anniversary of Academician S. M. Nikol′skiĭ [8].

2. Proofs.

PROOF OF THEOREM 2. We use the idea of representing an operator as the sum of "diagonal" operators by analogy to the linear case in [9] (see also [4, p. 464]).

Using standard arguments, we can equivalently restate the assertion of the theorem as follows: if V is a bounded positive polylinear operator acting from $\prod_{i=1}^n c_0(\rho(2^k))$ into $l_1(\rho(2^k))$ and from $\prod_{i=1}^n c_0(\rho(2^k)2^{-k})$ into $l_1(\rho(2^k)2^{-k})$ with the respective norms C_0 and C_1 then V acts boundedly from $\prod_{i=1}^n l_\infty$ into l_1, where $\rho(t) = \varphi(1,t)$ as above. Denote $\|V\| = \max(C_0, C_1)$.

The operator V is determined by the following sequence of infinite numeric "matrices":

$$\mathcal{V}^k = \left(v^k_{p_1,\dots,p_n}\right), \quad v^k_{p_1,\dots,p_n} \geq 0, \quad (p_1,\dots,p_n) \in \mathbf{Z}^n, \quad k \in \mathbf{Z}.$$

Indeed, suppose that $\{e_i\}_{i=-\infty}^{\infty}$ is the standard basis for the space c_0 and $a^i = \sum_{j=-\infty}^{\infty} a^i_j e_j$ $(1 \leq i \leq n)$ is a compactly-supported sequence. Then the linearity of V in each argument implies that

$$V(a^1,\dots,a^n) = (b_k)_{k=-\infty}^{\infty}, \quad b_k = \sum_{p_1=-\infty}^{\infty} \cdots \sum_{p_n=-\infty}^{\infty} a^1_{p_1} \dots a^n_{p_n} v^k_{p_1,\dots,p_n},$$

where $v^k_{p_1,\dots,p_n} = (V(e_{p_1},\dots,e_{p_n}))_k$.

Enumerate the system of the Rademacher functions on $[0,1]$ so that it constitute an infinite two-sided sequence $\{h_j(t)\}_{j=-\infty}^{\infty}$ with $h_0(t) = 1$. Given $m \in \mathbf{Z}$ and $t \in [0,1]$, define the sequence of "matrices"

$$\mathcal{V}^k_{m,t} = h_{k-m}(t)\left(h_{-p_1-\dots-p_n}(t) v^k_{p_1,\dots,p_n}\right), \quad (p_1,\dots,p_n) \in \mathbf{Z}^n, \quad k \in \mathbf{Z}.$$

Each of these sequences also determines a bounded polylinear operator from $\prod_{i=1}^n c_0(\rho(2^k))$ into $l_1(\rho(2^k))$ and from $\prod_{i=1}^n c_0(\rho(2^k)2^{-k})$ into $l_1(\rho(2^k)2^{-k})$ with the respective norms not exceeding $\|V\|$.

Averaging over all $t \in [0,1]$ with respect to the Lebesgue measure, we infer that a similar property is enjoyed by the operator V_m determined by the following sequence of "diagonal matrices":

$$\mathcal{V}_m = \left(v^k_{p_1,\dots,p_n} \delta_{p_n, k-m-p_1-\dots-p_{n-1}}\right), \quad (p_1,\dots,p_n) \in \mathbf{Z}^n,$$

where $\delta_{i,j}$ is the Kronecker symbol. Therefore, in view of the condition (*), for $m \geq 0$ we obtain

$$\|V\| \geq \|V_m\|_{\prod_{i=1}^n c_0(\rho(2^k)) \to l_1(\rho(2^k))} = \left\| \left(\frac{\rho(2^k)}{\rho(2^{p_1}) \dots \rho(2^{p_n})} v^k_{p_1,\dots,p_n} \delta_{p_n, k-m-p_1-\dots-p_{n-1}} \right) \right\|_{\prod_{i=1}^n c_0 \to l_1}$$

$$\geq K^{-n} \rho(2^m) \|V_m\|_{\prod_{i=1}^n c_0 \to l_1}.$$

By nonnegativity of $v^k_{p_1,\dots,p_n}$, the last inequality is equivalent to

$$\sum_{k=-\infty}^{\infty} \sum_{p_1=-\infty}^{\infty} \dots \sum_{p_{n-1}=-\infty}^{\infty} v^k_{p_1,\dots,p_{n-1},k-m-p_1-\dots-p_{n-1}} \leq \frac{\|V\| K^n}{\rho(2^m)}$$

or

$$\|V\|_{\prod_{i=1}^n l_\infty \to l_1} \leq \frac{\|V\| K^n}{\rho(2^m)} \quad (m \geq 0). \tag{6}$$

Estimating from below the norm of the operator V_m from $\prod_{i=1}^n c_0(\rho(2^k)2^{-k})$ into $l_1(\rho(2^k)2^{-k})$ in the case of $m < 0$, we obtain

$$\|V_m\|_{\prod_{i=1}^n l_\infty \to l_1} \leq \frac{2^m \|V\| K^n}{\rho(2^m)}. \tag{7}$$

Since $V = \sum_{m=-\infty}^{\infty} V_m$, the inequalities (6) and (7) imply that

$$\|V\|_{\prod_{i=1}^n l_\infty \to l_1} \leq \sum_{m=-\infty}^{\infty} \|V_m\|_{\prod_{i=1}^n l_\infty \to l_1} \leq C_1 \|V\|,$$

where

$$C_1 = \sum_{m=0}^{\infty} \left(\frac{1}{\rho(2^m)} + \frac{2^{-m}}{\rho(2^{-m})} \right) < \infty$$

by (**). Consequently, $V : \prod_{i=1}^n l_\infty \to l_1$ and Theorem 2 is proven.

PROOF OF THEOREM 1. Suppose that A is a bounded positive polylinear operator from $\prod_{i=1}^n X_0^i$ into Y_0 and from $\prod_{i=1}^n X_1^i$ into Y_1 with the respective norms C_0 and C_1 and put $\|A\| = \max(C_0, C_1)$.

If $x_i \in \varphi(X_0^i, X_1^i)$ then $x_i \in \mathrm{Orb}^+_{\vec{l}_\infty}(l_\infty(\varphi^*(1, 2^{-k})), \vec{X}^i)$ with $\vec{X}^i = \{X_0^i, X_1^i\}$ $(i = 1, 2, \dots, n)$ in view of (4). First suppose that there are $T_i \geq 0$, $T_i \in L(\vec{l}_\infty, \vec{X}^i)$, and $a_i \in l_\infty(\varphi^*(1, 2^{-k}))$ for which $x_i = T_i a^i$ $(i = 1, 2, \dots, n)$. As above, $\vec{l}_p = \{l_p, l_p(2^{-k})\}$.

Given a bounded positive linear operator Q from the pair $\vec{Y} = \{Y_0, Y_1\}$ into the pair $\vec{l}_1$, define the operator V:

$$V(\xi_1, \dots, \xi_n) = QA(T_1\xi_1, \dots, T_n\xi_n).$$

Then V is positive, polylinear, and acts boundedly from $\prod_{i=1}^n l_\infty$ into l_1 and from $\prod_{i=1}^n l_\infty(2^{-k})$ into $l_1(2^{-k})$. Moreover,

$$\|V\| = \max(\|V\|_{\prod_{i=1}^n l_\infty \to l_1}, \|V\|_{\prod_{i=1}^n l_\infty(2^{-k}) \to l_1(2^{-k})}) \leq \|Q\| \, \|A\| \prod_{i=1}^n \|T_i\|,$$

$$\|Q\| = \|Q\|_{\vec{Y}\to\vec{l}_1}, \quad \|T_i\| = \|T_i\|_{\vec{l}_\infty\to\vec{X}^i}.$$

By Theorem 2, V acts boundedly from $\prod_{i=1}^n l_\infty(\varphi^*(1,2^{-k}))$ into $l_1(\varphi^*(1,2^{-k}))$ with a norm not exceeding $C_1\|V\|$.

Suppose that $y = A(x_1,\dots,x_n) = A(T_1a_1,\dots,T_na_n)$. By the definition of Corb^+, we have

$$\|y\|_{\mathrm{Corb}^+} = \sup\{\|Qy\|_{\varphi,1} : \|Q\| \le 1\} = \sup\{\|V(a_1,\dots,a_n)\|_{\varphi,1} : \|Q\| \le 1\}$$
$$\le C_1\prod_{i=1}^n \|a_i\|_{\varphi,\infty} \sup\{\|V\| : \|Q\| \le 1\} \le C_1\|A\|\prod_{i=1}^n \|T_i\|\|a_i\|_{\varphi,\infty},$$

where $\|\cdot\|_{\varphi,p}$ stands for the norm in the space $l_p(\varphi^*(1,2^{-k}))$. Hence, owing to the embedding (5), we infer that the following inequality holds with some $C_2 > 0$:

$$\|y\|''_\varphi \le C_2\|A\|\prod_{i=1}^n \|T_i\|\,\|a_i\|_{\varphi,\infty}, \tag{8}$$

where $\|\cdot\|''_\varphi$ is the norm in the space $\Phi(Y''_0, Y''_1) \cap (Y_0 + Y_1)$.

In the general case

$$x_i = \sum_{j=1}^\infty T_i^j a_i^j, \tag{9}$$

where the series converges in $X_0^i + X_1^i$, $T_i^j \in L(\vec{l}_\infty, \vec{X})$, $a_i^j \in l_\infty(\varphi^*(1,2^{-k}))$ and, finally,

$$\sum_{j=1}^\infty \|T_i^j\|_{\vec{l}_\infty\to\vec{X}^i}\|a_i^j\|_{\varphi,\infty} < \infty \quad (i = 1,2,\dots,n).$$

For an arbitrary collection $(p_1,\dots,p_n) \in \mathbb{N}^n$ we put

$$y_{p_1,\dots,p_n} = A\big(T_1^{p_1}a_1^{p_1},\dots,T_n^{p_n}a_n^{p_n}\big).$$

Then, the linearity of T_i^j and the polylinearity of A yield

$$y = A(x_1,\dots,x_n) = \sum_{p_1=1}^\infty \cdots \sum_{p_n=1}^\infty y_{p_1,\dots,p_n}.$$

Applying (8) to $y_{p_1,\dots,p_n}$, we obtain

$$\|y_{p_1,\dots,p_n}\|''_\varphi \le C_2\|A\|\prod_{i=1}^n \|T_i^{p_i}\|\,\|a_i^{p_i}\|_{\varphi,\infty},$$

whence

$$\|y\|''_\varphi \le \sum_{p_1=1}^\infty \cdots \sum_{p_n=1}^\infty \|y_{p_1,\dots,p_n}\|''_\varphi \le C_2\|A\| \sum_{p_1=1}^\infty \cdots \sum_{p_n=1}^\infty \prod_{i=1}^n \|T_i^{p_i}\|\,\|a_i^{p_i}\|_{\varphi,\infty}$$
$$= C_2\|A\|\prod_{i=1}^n \left(\sum_{j=1}^\infty \|T_i^j\|\,\|a_i^j\|_{\varphi,\infty}\right).$$

Taking the greatest lower bounds on the right-hand side over the representations (9), from the definition of the norm in Orb^+ we derive

$$\|A(x_1,\dots,x_n)\|''_\varphi \le C_2\|A\|\prod_{i=1}^n \|x_i\|_{\mathrm{Orb}^+}.$$

Hence, A acts boundedly from $\prod_{i=1}^n \varphi(X_0^i, X_1^i)$ into $\varphi(Y_0'', Y_1'')\cap(Y_0+Y_1)$ and $\|A\|''_\varphi \le C_3\|A\|$ in view of (4). The theorem is proven.

Applications. Here we just scketch some possible applications of Theorem 1.

POLYLINEAR INTERPOLATION IN L_p-SPACES. Suppose that $w = w(t)$ is a positive measurable function on a space T with a measure μ. Then, as usual, $L_p(w)$ consists of all measurable functions $x = x(t)$ on T for which $w(t)x(t) \in L_p$ and $\|x\|_{L_p(w)} = \|wx\|_{L_p}$.

Since $\varphi(L_p(w^0), L_p(w^1)) = L_p(\varphi^*(w^0, w^1))$ [4, p. 459], Theorem 1 yields

Theorem 4. *If $\varphi \in \Phi$ satisfies the conditions $(*)$ and $(**)$ and a positive polylinear operator A acts boundedly from $\prod_{i=1}^n L_p(w_i^0)$ into $L_q(v_0)$ and from $\prod_{i=1}^n L_p(w_i^1)$ into $L_q(v_1)$ $(1 \le p, q \le \infty)$ then A acts boundedly from $\prod_{i=1}^n L_p(\varphi^*(w_i^0, w_i^1))$ into $L_q(\varphi^*(v_0, v_1))$.*

REMARK 5. In the case of $p = q = \infty$ a similar result was obtained in [10] for arbitrary bilinear (not necessarily positive) operators.

POLYLINEAR INTERPOLATION IN ORLICZ SPACES. Suppose that $N(t)$ is an increasing convex function on $[0,\infty)$ and $N(0) = 0$. The Orlicz space L_N^* consists of all measurable functions $x = x(t)$ on T such that

$$\int_T N(|x(t)|/u)\,d\mu < \infty$$

for some $u > 0$ with the norm

$$\|x\| = \inf\left\{u > 0 : \int_T N(|x(t)|/u)\,d\mu \le 1\right\}.$$

Consider a pair $\{L_{N_0}^*, L_{N_1}^*\}$ of Orlicz spaces. Given $\varphi \in \Phi$, denote by $N(t)$ the function such that

$$N^{-1}(t) = \varphi(N_0^{-1}(t), N_1^{-1}(t)),$$

where $N^{-1}(t)$ is the inverse function of N. Since $\varphi(L_{N_0}^*, L_{N_1}^*) = L_N^*$ [4, pp. 460–461] and $(L_N^*)'' = L_N^*$, from Theorem 1 we obtain the following natural polylinear interpolation theorem for Orlicz spaces:

Theorem 5. *If $\varphi \in \Phi$ satisfies the conditions $(*)$ and $(**)$ and a polylinear positive operator A acts boundedly from $\prod_{i=1}^n L_{N_i}^*$ into $L_{R_0}^*$ and from $\prod_{i=1}^n L_{M_i}^*$ into $L_{R_1}^*$ then A acts boundedly from $\prod_{i=1}^n L_{G_i}^*$ into L_R^*, where*

$$G_i^{-1}(t) = \varphi(N_i^{-1}(t), M_i^{-1}(t)) \ (i = 1,\dots,n), \quad R^{-1}(t) = \varphi(R_0^{-1}(t), R_1^{-1}(t)).$$

REMARK 6. In [10], a similar result was obtained for rearrangement invariant Marcinkiewicz spaces [11, p. 152] and arbitrary bilinear (not necessarily positive) operators.

CONVOLUTION OPERATORS AND TENSOR PRODUCTS. Denote by S the bilinear convolution operator for functions defined on $\mathbf{R}^m$:

$$S(x,y)(t) = \int_{\mathbf{R}^m} x(t-s)y(s)\,ds, \quad t \in \mathbf{R}^m.$$

Theorem 6. *If the operator S acts boundedly from $X_0^1 \times X_0^2$ into Y_0 and from $X_1^1 \times X_1^2$ into Y_1 (X_j^i and Y_j are* Bls *of functions defined on $\mathbf{R}^m$) and the function $\varphi \in \Phi$ satisfies the conditions $(*)$ and $(**)$ then S acts boundedly from $\varphi(X_0^1, X_1^1) \times \varphi(X_0^2, X_1^2)$ into $\varphi(Y_0'', Y_1'') \cap (Y_0 + Y_1)$.*

In study of some problems connected with the geometric properties of rearrangement invariant spaces [11] an important role is played by the tensor product operator

$$B(x,y)(s,t) = x(s)y(t),$$

where $x = x(s)$ and $y = y(t)$ are measurable functions on $[0,1]$ (see, for instance, [12, 13]).

Theorem 7. *If the operator B acts boundedly from $X_0^1 \times X_0^2$ into Y_0 and from $X_1^1 \times X_1^2$ into Y_1 (X_j^i and Y_j are rearrangement invariant spaces on $[0,1]$ and on $[0,1] \times [0,1]$) and the function $\varphi \in \Phi$ satisfies the conditions* (*) *and* (**), *then B acts boundedly from $\varphi(X_0^1, X_1^1) \times \varphi(X_0^2, X_1^2)$ into $\varphi(Y_0'', Y_1'') \cap (Y_0 + Y_1)$.*

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