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Interpolation of Bilinear Operators in Marcinkiewicz Spaces

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ABSTRACT. A theorem on interpolation of bilinear operators in symmetric Marcinkiewicz spaces is proved. It follows from the general bilinear results for the Peetre and Peetre–Gustavsson interpolation functors.

KEY WORDS: Marcinkiewicz spaces, Peetre interpolation functor, bilinear operators.

Since the functors of the real interpolation method whose parameters are "weighted" l-spaces interpolate bilinear operators [1], we can readily obtain the following interpolation theorem for the Lorentz spaces $\Lambda(\varphi)$ [2]. If T is a bilinear operator bounded from $\Lambda(\varphi_0)\times\Lambda(\psi_0)$ into $\Lambda(\theta_0)$ and from $\Lambda(\varphi_1)\times\Lambda(\psi_1)$ into $\Lambda(\theta_1)$, then for any quasiconcave function $\rho=\rho(t)$ the operator T is bounded from $\Lambda(\varphi_\rho)\times\Lambda(\psi_\rho)$ into $\Lambda(\theta_\rho)$, where

$$\varphi_\rho=\varphi_0\rho\biggl(\frac{\varphi_1}{\varphi_0}\biggr),\quad \psi_\rho=\psi_0\rho\biggl(\frac{\psi_1}{\psi_0}\biggr),\quad \theta_\rho=\theta_0\rho\biggl(\frac{\theta_1}{\theta_0}\biggr).$$

The main goal of this paper is to prove a similar theorem for Marcinkiewicz spaces. By analogy with the above-described case, it seems natural to use real interpolation functors with ℓ_∞ parameters. However, this would not yield the desired result even for a power law function $\rho(t)$ (see Theorem 3 in [1]). Moreover, note that the functions φ_i, ψ_i, and θ_i $(i=0,1)$ must satisfy certain conditions, since the bilinear interpolation theorem for Marcinkiewicz spaces is not valid in the general situation (see Remark 3 below). In the present paper, the theorem is proved under the condition that the dilatation exponents for these functions are nontrivial and turns out to be a consequence of the general bilinear interpolation theorems for the Peetre and Peetre–Gustavsson interpolation functors [3, 4].

The paper is organized as follows. In the first section, we give some definitions and notation adopted in interpolation theory and playing an important part in the sequel. In §2 we obtain the theorems for the Peetre and Peetre–Gustavsson interpolation functors. Finally, §3 deals with applications: here we treat interpolation of bilinear operators in ℓ_∞ sequence spaces and in Marcinkiewicz spaces.

§1. Introduction

Let us recall some definitions from the interpolation theory of linear operators (see [2, 5] for details).

A triple (X_0,X_1,X) of Banach spaces is said to be ***interpolational with respect to a triple*** (Y_0,Y_1,Y) if any linear operator T continuous from X_i into Y_i $(i=0,1)$ is necessarily continuous from X into Y. By an ***interpolation functor*** we mean a functor F from the category of Banach couples into the category of Banach spaces such that for any couples $\vec X=(X_0,X_1)$ and $\vec Y=(Y_0,Y_1)$ the triple $(X_0,X_1,F(\vec X))$ is interpolational with respect to $(Y_0,Y_1,F(\vec Y))$.

In what follows, $\rho=\rho(t)$ is a positive quasiconcave function. This means that $\rho(t)$ is increasing and $\rho(t)/t$ is decreasing for $t>0$; the dilatation function is defined by

$$\mathcal{M}(t)=\sup_{s>0}\frac{\rho(st)}{\rho(s)}\qquad(t>0).$$

The numbers

$$\gamma_\rho=\lim_{t\to0+}\frac{\ln\mathcal{M}(t)}{\ln t},\qquad \delta_\rho=\lim_{t\to\infty}\frac{\ln\mathcal{M}(t)}{\ln t},$$

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are well defined [2] and are called the *upper* and the *lower dilatation exponent* of ρ, respectively. If ρ is concave, then $0 < \gamma_\rho \le \delta_\rho \le 1$.

Let us define the Peetre interpolation functor $\langle \vec{X} \rangle_\rho$ [3] and the Peetre–Gustavsson interpolation functor $\langle \vec{X}, \rho \rangle$ [4], which were introduced in connection with the interpolation of Orlicz spaces.

The space $\langle \vec{X} \rangle_\rho$ contains all $x \in X_0 + X_1$ that admit a representation of the form

$$x = \sum_{n=-\infty}^{\infty} x_n \qquad (\text{convergence in } X_0 + X_1), \qquad \text{where} \quad x_n \in X_0 \cap X_1 \tag{1}$$

and the sequences $\{x_n/\rho(2^n)\}_{n=-\infty}^{\infty}$ and $\{2^n x_n/\rho(2^n)\}_{n=-\infty}^{\infty}$ unconditionally converge in X_0 and X_1, respectively. The norm on this space is defined by $\|x\|_\rho = \inf C$, where

$$C = \max\left\{ \sup_{\varepsilon_n = \pm 1} \left\| \sum_{n=-\infty}^{\infty} \frac{\varepsilon_n x_n}{\rho(2^n)} \right\|_{X_0}, \sup_{\varepsilon_n = \pm 1} \left\| \sum_{n=-\infty}^{\infty} \frac{\varepsilon_n 2^n x_n}{\rho(2^n)} \right\|_{X_1} \right\} \tag{2}$$

and the infimum is taken over all representations of x. By $\langle \vec{X}, \rho \rangle$ we denote the set of all $x \in X_0 + X_1$ admitting a representation of the form (1), where $x_n \in X_0 \cap X_1$ and the sequences $\{x_n/\rho(2^n)\}_{n=-\infty}^{\infty}$ and $\{2^n x_n/\rho(2^n)\}_{n=-\infty}^{\infty}$ weakly unconditionally converge in X_0 and X_1, respectively [4]. The norm on $\langle \vec{X}, \rho \rangle$ is defined by $\|x\| = \inf C$, where

$$C = \max\left\{ \sup_{\substack{\varepsilon_k = \pm 1 \\ F \subset \mathbb{Z}}} \left\| \sum_{k \in F} \frac{\varepsilon_k x_k}{\rho(2^k)} \right\|_{X_0}, \sup_{\substack{\varepsilon_k = \pm 1 \\ F \subset \mathbb{Z}}} \left\| \sum_{k \in F} \frac{\varepsilon_k 2^k x_k}{\rho(2^k)} \right\|_{X_1} \right\}$$

($F \subset \mathbb{Z}$ is a finite set) and the infimum is taken over all representations of x.

Finally, let us recall the definitions of real interpolation functors with ℓ_∞ parameters. For a Banach couple $\vec{X} = (X_0, X_1)$ and for $t > 0$, we define the Peetre $\mathcal{K}$- and $\mathcal{F}$-functionals [6] by the formulas

$$\mathcal{K}(t, x; \vec{X}) = \inf\{ \|x_0\|_{X_0} + t\|x_1\|_{X_1};\, x_0 + x_1 = x, x_i \in X_i \} \qquad (x \in X_0 + X_1);$$
$$\mathcal{F}(t, x; \vec{X}) = \max\{ \|x\|_{X_0}, t\|x\|_{X_1} \} \qquad (x \in X_0 \cap X_1).$$

The space $(X_0, X_1)^{\mathcal{K}}_{\rho,\infty}$ of the $\mathcal{K}$-method consists of all $x \in X_0 + X_1$ such that the norm

$$\|x\| = \sup_j \frac{\mathcal{K}(2^j, x; \vec{X})}{\rho(2^j)}$$

is finite. The space $(X_0, X_1)^{\mathcal{F}}_{\rho,\infty}$ of the $\mathcal{F}$-method consists of all $x \in X_0 + X_1$ admitting a representation of the form (1), where $x_n \in X_0 \cap X_1$, and the norm is defined as follows:

$$\|x\| = \inf \sup_n \frac{\mathcal{F}(2^n, x_n; \vec{X})}{\rho(2^n)},$$

where the infimum is taken over all representations of x.

§2. General interpolation theorems for bilinear operators

Theorem 1. *Let $\rho = \rho(t)$ be a quasiconcave function on the semiaxis $(0, \infty)$ satisfying the conditions*

1) $0 < \gamma_\rho \le \delta_\rho < 1$;
2) *there exists an $\alpha > 0$ such that for each $u > 0$ and $v > 0$ the inequality*

$$\rho(u) \cdot \rho(v) \le \alpha \rho(u, v) \tag{3}$$

holds.

Suppose that $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$, and $\vec{Z} = (Z_0, Z_1)$ are arbitrary Banach couples and T is a bilinear operator continuous from $X_0 \times Y_0$ into Z_0 and from $X_1 \times Y_1$ into Z_1. Then T extends to a bilinear operator continuous from $\langle \vec{X} \rangle_\rho \times \langle \vec{Y} \rangle_\rho$ into $(Z_0, Z_1)^{\mathcal{K}}_{\rho,\infty}$.

We need the following auxiliary assertion.

Lemma 1. *Suppose that X and Y are Banach spaces, $F \subset \mathbb{Z}$ is finite, $\{x_n\} \subset X$, $\{y_n\} \subset Y$, and*

$$C_1 = \sup_{\varepsilon_n=\pm1} \Big\| \sum_{n\in F} \varepsilon_n x_n \Big\|_X, \qquad C_2 = \sup_{\delta_m=\pm1} \Big\| \sum_{m\in F} \delta_m y_m \Big\|_Y.$$

If T is a bilinear operator continuous from $X \times Y$ into a Banach space Z, then

$$\Big\| \sum_{n\in F} T(x_n, y_n) \Big\|_Z \le C_1 C_2 \|T\|. \tag{4}$$

Proof. First, since T is continuous and bilinear, it follows that

$$\Big\| \sum_{n\in F} \sum_{m\in F} T(\varepsilon_n x_n, \delta_m y_m) \Big\|_Z = \Big\| T\Big(\sum_{n\in F} \varepsilon_n x_n \sum_{m\in F} \delta_m y_m \Big) \Big\|_Z \le \|T\| C_1 C_2. \tag{5}$$

Let $\min F = s$. By applying (5) first with $\delta_s = \varepsilon_s = 1$ and then with $\delta_s = \varepsilon_s = -1$, we obtain

$$\Big\| T(x_s, y_s) + \sum_{m\in G} T(x_s, \delta_m y_m) + \sum_{n\in G} T(\varepsilon_n x_n, y_s) + \sum_{n\in G} \sum_{m\in G} T(\varepsilon_n x_n, \delta_m y_m) \Big\|_Z \le \|T\| C_1 C_2,$$

$$\Big\| T(x_s, y_s) + \sum_{m\in G} T(x_s, \delta_m y_m) - \sum_{n\in G} T(\varepsilon_n x_n, y_s) + \sum_{n\in G} \sum_{m\in G} T(\varepsilon_n x_n, \delta_m y_m) \Big\|_Z \le \|T\| C_1 C_2,$$

where $G = F \setminus \{s\}$. By passing to the arithmetic mean of the left-hand sides of the last inequalities, we arrive at the relation

$$\Big\| T(x_s, y_s) + \sum_{n\in G} \sum_{m\in G} T(\varepsilon_n x_n, \delta_m y_m) \Big\|_Z \le \|T\| C_1 C_2.$$

A successive application of this trick yields inequality (4). The proof of the lemma is complete. □

Proof of Theorem 1. Let $x \in \langle \vec{X} \rangle_\rho$ and $y \in \langle \vec{Y} \rangle_\rho$. Then

$$x = \sum_{k=-\infty}^{\infty} u_k \qquad (u_k \in X_0 \cap X_1, \quad \text{convergence in} \quad X_0 + X_1),$$

and moreover, the sequences $\{u_k/\rho(2^k)\}$ and $\{2^k u_k/\rho(2^k)\}$ converge unconditionally in X_0 and X_1, respectively. Similarly,

$$y = \sum_{i=-\infty}^{\infty} v_i \qquad (v_i \in Y_0 \cap Y_1, \quad \text{convergence in} \quad Y_0 + Y_1),$$

and the sequences $\{v_i/\rho(2^i)\}$ and $\{2^i v_i/\rho(2^i)\}$ unconditionally converge in Y_0 and Y_1, respectively. Let us write

$$w_m = \sum_{k=-\infty}^{\infty} T(u_k, v_{m-k})$$

and prove that the sequence $\{T(u_k, v_{m-k})\}_{k=-\infty}^{\infty}$ for each $m \in \mathbb{Z}$ converges unconditionally in Z_0 and Z_1. We use the representation

$$\frac{w_m}{\rho(2^m)} = \sum_{k=-\infty}^{\infty} a_{m,k} T\Big(\frac{u_k}{\rho(2^k)}, \frac{v_{m-k}}{\rho(2^{m-k})} \Big), \qquad \text{where} \quad a_{m,k} = \frac{\rho(2^k)\rho(2^{m-k})}{\rho(2^m)}.$$

Note that by (3) we have

$$a_{m,k} \le \alpha. \tag{6}$$

By Lemma 1, for arbitrary finite $F \subset \mathbb{Z}$ we have

$$\left\|\sum_{k\in F} T\left(\frac{u_k}{\rho(2^k)}, \frac{v_{m-k}}{\rho(2^{m-k})}\right)\right\|_{Z_0} \le \|T\| \sup_{\varepsilon_k=\pm 1}\left\|\sum_{k\in F}\frac{\varepsilon_k u_k}{\rho(2^k)}\right\|_{X_0} \sup_{\delta_k=\pm 1}\left\|\sum_{k\in F}\frac{\delta_k v_{m-k}}{\rho(2^{m-k})}\right\|_{Y_0}. \tag{7}$$

It follows from (7) by virtue of the unconditional convergence of $\{u_k/\rho(2^k)\}_{k=-\infty}^{\infty}$ and $\{v_i/\rho(2^i)\}_{i=-\infty}^{\infty}$ in X_0 and Y_0, respectively, that $\{T(u_k/\rho(2^k), v_{m-k}/\rho(2^{m-k}))\}_{k=-\infty}^{\infty}$ unconditionally converges in Z_0. In view of (6), $\{a_{m,k}T(u_k/\rho(2^k), v_{m-k}/\rho(2^{m-k}))\}_{k=-\infty}^{\infty}$ also converges unconditionally in Z_0; moreover, by applying (7) once more, we obtain

$$\left\|\frac{w_m}{\rho(2^m)}\right\|_{Z_0} \le \alpha\|T\|C_1C_2, \qquad \|T\| = \max_{i=0,1}\|T\|_{X_i\times Y_i\to Z_i}, \tag{8}$$

where C_1 and C_2 are defined in the same way as in (2). In particular, it follows from (8) that for arbitrary finite $F \subset \mathbb{Z}$ and any $m \in \mathbb{Z}$ we have

$$\left\|\sum_{k\in F} T(u_k, v_{m-k})\right\|_{Z_0} \le \alpha\|T\|C_1C_2\rho(2^m). \tag{9}$$

Similarly, we can prove the unconditional convergence of the sequence

$$\{a_{m,k}T(2^k u_k/\rho(2^k), 2^{m-k}v_{m-k}/\rho(2^{m-k}))\}_{k=-\infty}^{\infty}$$

in Z_1 and the estimates

$$\left\|\frac{2^m w_m}{\rho(2^m)}\right\|_{Z_1} \le \alpha\|T\|C_1C_2, \tag{10}$$

$$\left\|\sum_{k\in F} T(v_k, v_{m-k})\right\|_{Z_1} \le \alpha\|T\|C_1C_2\rho(2^m)2^{-m}. \tag{11}$$

Consequently, $w_m \in Z_0 + Z_1$ and, by (8)-(11),

$$\|w_m\|_{Z_0+Z_1} = \mathcal{K}(1, w_m; \vec{Z}) \le \alpha\|T\|C_1C_2\min\{\rho(2^m), \rho(2^m)2^{-m}\}, \tag{12}$$

$$\left\|\sum_{k\in F} T(u_k, v_{m-k})\right\|_{Z_0+Z_1} \le \alpha\|T\|C_1C_2\min\{\rho(2^m), \rho(2^m)2^{-m}\} \tag{13}$$

for any finite $F \subset \mathbb{Z}$ and any $m \in \mathbb{Z}$.

The series $\sum_{m=-\infty}^{\infty} w_m$, whose sum will be denoted by S, absolutely converges in $Z_0 + Z_1$, since it follows from (12) that

$$\sum_{m=-\infty}^{\infty} \|w_m\|_{Z_0+Z_1} \le \alpha\|T\|C_1C_2C_\rho,$$

where, in view of condition 1) for ρ, we have

$$C_\rho = \sum_{m=-\infty}^{\infty} \min\{\rho(2^m), \rho(2^m)2^{-m}\} < \infty.$$

Moreover, from (8) and (10) we obtain

$$\mathcal{F}(2^m, w_m\,;\,\vec{Z}) = \max\{\|w_m\|_{Z_0}, 2^m\|w_m\|_{Z_1}\} \leq \alpha\|T\|C_1C_2\rho(2^m).$$

Since $0 < \gamma_\rho \leq \delta_\rho < 1$, we have the equality (isomorphism) [7, 8]

$$(Z_0, Z_1)^{\mathcal{K}}_{\rho,\infty} = (Z_0, Z_1)^{\mathcal{F}}_{\rho,\infty}. \tag{14}$$

Thus, $S \in (Z_0, Z_1)^{\mathcal{K}}_{\rho,\infty}$, and the norm of S in that space can be estimated as follows:

$$\|S\|^{\mathcal{K}}_{\rho,\infty} \leq \alpha\beta\|T\|C_1C_2, \tag{15}$$

where β is a constant related to the isomorphism (14).

Let us show that S is independent of the sequences $\{u_k\}$ and $\{v_i\}$, but is determined by the elements x and y alone. To this end, let us prove that

$$S = \lim_{\substack{M\to\infty\\ N\to\infty}} \sum_{\substack{|k|\leq M\\ |i|\leq N}} T(u_k, v_i) \qquad \text{(convergence in} \quad Z_0 + Z_1). \tag{16}$$

In the following we use the norm of the space $Z_0 + Z_1$. For any $\varepsilon > 0$, using the absolute convergence of the series $\sum_{m=-\infty}^{\infty} w_m$ and inequality (13), we can find an index m_0 such that

$$\Big\|\sum_{|m|\leq m_0} w_m - S\Big\| \leq \frac{\varepsilon}{2}, \tag{17}$$

$$\sum_{|m|>m_0}\Big\|\sum_{k\in F_m} T(u_k, v_{m-k})\Big\| \leq \frac{\varepsilon}{4}, \tag{18}$$

where F_m is an arbitrary finite subset of $\mathbb{Z}$. Next, there exists a k_0 such that for arbitrary $k_m \geq k_0$ ($|m| \leq m_0$) we have

$$\sum_{|m|\leq m_0}\Big\|\sum_{|k|>k_m} T(u_k, v_{m-k})\Big\| \leq \frac{\varepsilon}{4}. \tag{19}$$

Let $M > k_0$ and $N > m_0 + k_0$ be integers. We introduce the sets

$$P = \big\{(k,i) \in \mathbb{Z}^2 : |k| \leq M, |i| \leq N\big\}, \qquad Q = \big\{(k,i) \in \mathbb{Z}^2 : |k| \leq k, |i+k| \leq m_0\big\}.$$

Note that the conditions imposed on M and N imply $Q \subset P$. Let $F_m = \big\{k \in \mathbb{Z} : (k, m-k) \in P\big\}$ and $k_m = \max F_m$. It follows from the embedding $Q \subset P$ that for $|m| \leq m_0$ one has $k_m \geq k_0$, and consequently, the above-chosen F_m and k_m satisfy inequalities (18) and (19). A straightforward verification shows that

$$\sum_{\substack{|k|\leq M\\ |i|\leq N}} T(u_k, v_i) = \sum_{|m|\leq m_0} w_m + \sum_{|m|>m_0}\sum_{k\in F_m} T(u_k, v_{m-k}) - \sum_{|m|\leq m_0}\sum_{|k|>k_m} T(u_k, v_{m-k}).$$

Finally, by applying (17)– (19), we obtain

$$\Big\|\sum_{\substack{|k|\leq M\\ |i|\leq N}} T(u_k, v_i) - S\Big\| \leq \varepsilon$$

provided that $M > k_0$ and $N > m_0 + k_0$, which proves (16).

Now suppose that

$$x = \sum_{k=-\infty}^{\infty} u'_k \qquad (u'_k \in X_0 \cap X_1, \quad \text{convergence in} \quad X_0 + X_1)$$

and $\{u'_k\}$ satisfies the same conditions as $\{u_k\}$. We write

$$w'_m = \sum_{k=-\infty}^{\infty} T(u'_k, v_{m-k}) \quad (m \in \mathbb{Z}), \qquad S' = \sum_{m=-\infty}^{\infty} w'_m,$$

where $\{v_i\}$ is the same as above. Then, as was already indicated, we have

$$S' = \lim_{\substack{M\to\infty \\ N\to\infty}} \sum_{\substack{|k|\le M \\ |i|\le N}} T(u'_k, v_i) \qquad (\text{convergence in} \quad Z_0 + Z_1).$$

For $v \in Y_0 \cap Y_1$, the linear operator $A_v u = T(u, v)$ is continuous in $X_0 + X_1$; hence, by passing from the double limit to an iterated limit, we obtain

$$S = \lim_{N\to\infty} \sum_{|i|\le N} \lim_{M\to\infty} T\left(\sum_{|k|\le M} u'_k, v_i \right) = \lim_{N\to\infty} \sum_{|i|\le N} T(x, v_i) = S.$$

Similarly, the passage to another representation with respect to the second variable can be performed, which does not affect the value of S.

We see that the sum depends on x and y alone; hence, it will be denoted by $T(x, y)$. Using the absolute convergence of the series $\sum_{m=-\infty}^{\infty} w_m$, it is easy to prove that $T(x, y)$ is bilinear. It follows from inequality (15) that

$$\|T(x, y)\|^{\mathcal{K}}_{\rho,\infty} \le \alpha\beta\|T\|C_1C_2,$$

whence, by the definition of the norm in the spaces $\langle\vec{X}\rangle_\rho$, we obtain

$$\|T(x, y)\|^{\mathcal{K}}_{\rho,\infty} \le \alpha\beta\|T\|\, \|x\|_\rho \|y\|_\rho.$$

Thus, the norm of

$$T\colon \langle\vec{X}\rangle_\rho + \langle\vec{Y}\rangle_\rho \to (Z_0, Z_1)^{\mathcal{K}}_{\rho,\infty}$$

does not exceed $\alpha\beta\|T\|$, where $\|T\| = \max_{i=0,1} \|T\|_{X_i\times Y_i\to Z_i}$. The proof of the theorem is complete. $\square$

A similar argument allows us to prove the corresponding theorem for the Peetre–Gustavsson interpolation functor $\langle\,\cdot\,, \rho\rangle$.

Theorem 2. *Let $\rho = \rho(t)$ be a quasiconcave function on $(0, \infty)$ satisfying conditions 1) and 2) in Theorem 1. Suppose that T is a bilinear operator continuous from $X_0 \times Y_0$ into Z_0 and from $X_1 \times Y_1$ into Z_1. Then T extends to be a bilinear operator continuous from $\langle\vec{X}, \rho\rangle \times \langle\vec{Y}, \rho\rangle$ into $(Z_0^{**}, Z_1^{**})^{\mathcal{K}}_{\rho,\infty}$.*

Corollary 1. *If Z_0 and Z_1 are reflexive spaces and the assumptions of Theorem 2 are satisfied, then T extends to be a bilinear operator continuous from $\langle\vec{X}, \rho\rangle \times \langle\vec{Y}, \rho\rangle$ into $(Z_0, Z_1)^{\mathcal{K}}_{\rho,\infty}$.*

§3. Specific interpolation theorems

a) **Sequence spaces.** For a nonnegative number sequence $w = \{w_j\}_{j=-\infty}^{\infty}$, by $\ell_\infty(w)$ we denote the space ℓ_∞ with "weight" w; the norm on this space is defined by

$$\big\|\{a_j\}\big\|_{\ell_\infty(w)} = \big\|\{a_j w_j\}\big\|_{\ell_\infty} = \sup_{j\in\mathbb{Z}} |a_j w_j|.$$

We define the space $c_0(w)$ in a similar way.

A sequence $\{t_n\}_{n=-\infty}^{\infty}$ is said to be *sparse for the function* $\rho = \rho(t)$ if for any $A, B > 0$ the set of $n \in \mathbb{Z}$ such that

$$A \le \rho(t_n) \le B \qquad \text{or} \qquad A \le \rho(t_n)t_n^{-1} \le B$$

is finite [7].

In the following, if $\rho(t)$ is a quasiconcave function and $f^0 = \{f_i^0\}$ and $f^1 = \{f_i^1\}$ are two-way positive number sequences, we define a function $\rho^*(t)$ and a sequence f_ρ by setting

$$\rho^*(t) = \frac{1}{\rho(t^{-1})}, \qquad f_\rho = f^0 \rho\left(\frac{f^1}{f^0}\right).$$

Theorem 3. *Let a bilinear operator T be continuous from $c_0(u^0) \times c_0(v^0)$ into $\ell_\infty(w^0)$ and from $c_0(u^1) \times c_0(v^1)$ into $\ell_\infty(w^1)$. If a quasiconcave function $\rho = \rho(t)$ satisfies conditions 1) and 2) of Theorem 1, then T extends to be a bilinear operator continuous from $c_0(u_{\rho^*}) \times c_0(v_{\rho^*})$ into $\ell_\infty(w_{\rho^*})$.*

Proof. For arbitrary weight sequences $f^0 = \{f_n^0\}$ and $f^1 = \{f_n^1\}$, we set

$$\vec{c}_0(f) = \{c_0(f^0), c_0(f^1)\}, \qquad \vec{\ell}_\infty(f) = \{\ell_\infty(f^0), \ell_\infty(f^1)\}.$$

Then [7; 9, p. 461])

$$\langle \vec{c}_0(u)\rangle_\rho = c_0(u_{\rho^*}), \qquad \langle \vec{c}_0(v)\rangle_\rho = c_0(v_{\rho^*}).$$

Hence, by Theorem 1 we have

$$T\colon c_0(u_{\rho^*}) \times c_0(v_{\rho^*}) \to \big(\ell_\infty(w^0), \ell_\infty(w^1)\big)^{\mathcal{K}}_{\rho,\infty}.$$

On the other hand, by [9, p. 422] we can write

$$\big(\ell_\infty(w^0), \ell_\infty(w^1)\big)^{\mathcal{K}}_{\rho,\infty} = \ell_\infty(w_{\rho^*}). \tag{20}$$

Now we obtain the desired assertion by applying (20). □

Theorem 4. *Suppose that a bilinear operator T is continuous from $c_0(u^0) \times c_0(v^0)$ into $c_0(w^0)$ and from $c_0(u^1) \times c_0(v^1)$ into $c_0(w^1)$, and moreover, the sequences $\{u_n^0/u_n^1\}$ and $\{v_n^0/v_n^1\}$ are sparse for a quasiconcave function $\rho(t)$ satisfying conditions 1), 2) of Theorem 1. Then T extends to a bilinear operator continuous from $\ell_\infty(u_{\rho^*}) \times \ell_\infty(v_{\rho^*})$ into $\ell_\infty(w_{\rho^*})$.*

Proof. Under the assumptions of the theorem, we have [7; 9, p. 461]

$$\langle \vec{c}_0(u), \rho\rangle = \ell_\infty(u_{\rho^*}), \qquad \langle \vec{c}_0(v), \rho\rangle = \ell_\infty(v_{\rho^*}).$$

Since $c_0^{**} = \ell_\infty$, by Theorem 2 we have

$$T\colon \ell_\infty(u_{\rho^*}) \times \ell_\infty(v_{\rho^*}) \to \big(\ell_\infty(w^0), \ell_\infty(w^1)\big)^{\mathcal{K}}_{\rho,\infty}.$$

It remains to use Eq. (20) from the proof of the preceding theorem. □

Remark 1. Since $0 < \gamma_\rho \le \delta_\rho < 1$, we see that, say, the sequence $\{u_n^0/u_n^1\}$ will be sparse with respect to ρ if at least one of the following conditions is satisfied:

$$\lim_{n\to-\infty} \frac{u_n^0}{u_n^1} = \lim_{n\to+\infty} \frac{u_n^1}{u_n^0} = 0 \qquad \text{or} \qquad \lim_{n\to-\infty} \frac{u_n^1}{u_n^0} = \lim_{n\to+\infty} \frac{u_n^0}{u_n^1} = 0.$$

b) **Function spaces.** Recall that for a positive quasiconcave function φ on $(0,\infty)$, the *Marcinkiewicz space* $M(\varphi)$ is the Banach space of measurable functions on $(0,\infty)$ with finite norm

$$\|x\|_{M(\varphi)} = \sup_{t>0}[\varphi(t)]^{-1}\int_0^t x^*(s)\,ds.$$

By $M^0(\varphi)$ we denote the set of all $x \in M(\varphi)$ such that

$$\lim_{h\to 0,\infty}[\varphi(h)]^{-1}\int_0^h x^*(t)\,dt = 0.$$

Let a function f be defined on $(0,\infty)$, and let $f \geq 0$. Then by $\ell_\infty(f)$ we denote the space ℓ_∞ with "weight" $\{f(2^j)\}_{j=-\infty}^{\infty}$. In a similar way, the space $c_0(f)$ is defined.

Theorem 5. *Let a bilinear operator T be continuous from $M^0(\psi_0)\times M^0(\varphi_0)$ into $M^0(\theta_0)$ and from $M^0(\psi_1)\times M^0(\varphi_1)$ into $M^0(\theta_1)$, and let the functions $\psi_i=\psi_i(t)$, $\varphi_i=\varphi_i(t)$, and $\theta_i=\theta_i(t)$ $(i=0,1)$ possess the following properties:*

1) $0<\gamma_{\psi_i}\leq\delta_{\psi_i}<1$, $0<\gamma_{\varphi_i}\leq\delta_{\varphi_i}<1$, $0<\gamma_{\theta_i}\leq\delta_{\theta_i}<1$ $(i=0,1)$;
2) *the sequences $\{\psi_1(2^j)/\psi_0(2^j)\}$ and $\{\varphi_1(2^j)/\varphi_0(2^j)\}$ are sparse for any quasiconcave function $\rho(t)$ satisfying conditions* 1), 2) *of Theorem* 1.

Then T extends to a bilinear operator continuous from $M(\psi_\rho)\times M(\varphi_\rho)$ into $M(\theta_\rho)$.

Proof. For quasiconcave functions $f_0=f_0(t)$ and $f_1=f_1(t)$, set

$$\vec{M}^0(f)=\{M^0(f_0),M^0(f_1)\},\qquad \vec{M}(f)=\{M(f_0),M(f_1)\}.$$

It is known [2, 3] that for the Lebesgue spaces L_1 and L_∞ on the semiaxis $(0,\infty)$ and for $x\in L_1+L_\infty$ one has

$$\mathcal{K}(t,x;L_1,L_\infty)=\int_0^t x^*(s)\,ds.$$

Hence, for an arbitrary quasiconcave function f, we can write

$$M(f)=(L_1,L_\infty)^{\mathcal{K}}_{\ell_\infty(1/f)},\qquad M^0(f)=(L_1,L_\infty)^{\mathcal{K}}_{c_0(1/f)}.$$

By virtue of the conditions imposed on the dilatation exponents of ψ_i, φ_i, and θ_i $(i=0,1)$, it follows from the reiteration theorem [8, 10] that

$$\langle\vec{M}^0(\psi),\rho\rangle=(L_1,L_\infty)^{\mathcal{K}}_{\langle\vec{c}_0(1/\psi),\rho\rangle},\qquad \langle\vec{M}^0(\varphi),\rho\rangle=(L_1,L_\infty)^{\mathcal{K}}_{\langle\vec{c}_0(1/\varphi),\rho\rangle},$$
$$\big(\vec{M}(\theta)\big)^{\mathcal{K}}_{\rho,\infty}=(L_1,L_\infty)^{\mathcal{K}}_{(\vec{\ell}(1/\theta))^{\mathcal{K}}_{\rho,\infty}}.$$

As was indicated in the proof of Theorems 3 and 4,

$$\left\langle\vec{c}_0\Big(\frac{1}{\psi}\Big),\rho\right\rangle=\ell_\infty\Big(\frac{1}{\psi_\rho}\Big),\qquad \left\langle\vec{c}_0\Big(\frac{1}{\varphi}\Big),\rho\right\rangle=\ell_\infty\Big(\frac{1}{\varphi_\rho}\Big),\qquad \left(\vec{\ell}_\infty\Big(\frac{1}{\theta}\Big)\right)^{\mathcal{K}}_{\rho,\infty}=\ell_\infty\Big(\frac{1}{\theta_\rho}\Big).$$

Thus, in view of the formula $\big(M^0(\theta)\big)^{**}=M(\theta)$ [2, pp. 153–159], we obtain the assertion of Theorem 2 from Theorem 5. $\square$

Remark 2. From Remark 1 one readily obtains sufficient conditions for the sequences $\{\psi_1(2^j)/\psi_0(2^j)\}$ and $\{\varphi_1(2^j)/\varphi_0(2^j)\}$ to be sparse.

Remark 3. The condition that the dilatation exponents of φ_i, ψ_i, and θ_i are nontrivial is essential. Indeed, consider the tensor product operator

$$B(x, y) = x(s)y(t).$$

Then $B\colon L_1 \times L_1 \to L_1$ and $B\colon L_\infty \times L_\infty \to L_\infty$. The spaces L_1 and L_∞ are the "extreme" Marcinkiewicz spaces; namely, $L_1 = M(1)$ and $L_\infty = M(t)$. Should Theorem 5 be valid in this case, the operator B would be bounded from $M(t^\theta) \times M(t^\theta)$ into $M(t^\theta)$ $(0 < \theta < 1)$, which is not the case [11].

Remark 4. We can apply Theorem 5 to specific operators such as the tensor product operator B (see the preceding remark) or the convolution operator

$$S(x, y)(t) = \int_0^t x(s)y(t-s)\,ds.$$

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