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BRIEF COMMUNICATIONS

On the Multiplier of a Rearrangement Invariant Space with Respect to the Tensor Product

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Let $x = x(s)$ and $y = y(t)$ be measurable functions on $I = [0, 1]$. We introduce the bilinear operator

$$B(x, y)(s, t) = x \otimes y(s, t) = x(s)y(t), \qquad (s, t) \in I \times I.$$

Recall that a Banach space E of measurable functions on I is said to be *rearrangement invariant* (*r.i.*) if for any $y \in E$ the relation $x^*(t) \le y^*(t)$ (where $z^*(t)$ is the nonincreasing rearrangement of the function $|z(u)|$ [1, p. 93]) implies the relations $x \in E$ and $\|x\| \le \|y\|$.

If E is an r.i. space on I, then the corresponding r.i. space $E(I \times I)$ on the square consists of all measurable functions $x = x(s, t)$ on $I \times I$ such that $x^* \in E$ and $\|x\|_{E(I\times I)} = \|x^*\|_E$.

If X, Y, and Z are r.i. spaces on I, then the operator B from $X \times Y$ into $Z(I \times I)$ is continuous if and only if the embedding of the projective tensor product $X \otimes Y$ in Z is continuous [2, p. 51]. For this reason, B is called the tensor product operator.

Studying the operator B is important because it naturally arises in solving a number of problems related to geometric properties of r.i. spaces (e.g., see [3, p. 169–171; 4, 5]).

The majority of papers on the tensor product operator for r.i. spaces are devoted to studying continuity conditions for this operator in special classes of these spaces, namely, in Lorentz, Marcinkiewicz, Orlicz spaces, etc. [2, 6–9]. In this note we consider arbitrary r.i. spaces. Let us introduce the main notion of multiplier of an r.i. space with respect to the operator B.

Let E be an r.i. space on I. By the *multiplier* of this space we mean the set $M(E)$ of all measurable functions $x = x(s)$ on I such that $x \otimes y \in E(I \times I)$ for an arbitrary function $y \in E$. Then $M(E)$ is a linear space, which is an r.i. space on I with respect to the norm

$$\|x\|_{M(E)} = \sup\{\|x \otimes y\|_{E(I\times I)};\ \|y\|_E \le 1\}.$$

It is clear that we always have $M(E) \subset E$.

Consider an example. Let E be the Lorentz space $\Lambda(\phi)$, where $\phi(u)$ is a nonnegative increasing concave function on $(0, 1]$. Recall that this space consists of all measurable functions $x = x(s)$ such that

$$\|x\|_{\Lambda(\phi)} = \int_0^1 x^*(s)\, d\phi(s) < \infty.$$

Assume that for the dilation function

$$\mathscr{M}_\phi(v) = \sup\{\phi(tv)/\phi(t),\ 0 < t \le \min(1, 1/v)\}$$

we have

$$\mathscr{M}_\phi(v) = \lim_{t\to t_0} \phi(tv)/\phi(t),$$

where $t_0 \in [0, 1]$ does not depend on $v \in [0, 1]$. Simple calculations show that $M(\Lambda(\phi)) = \Lambda(\mathscr{M}_\phi)$.

Let $\chi_e(s) = 1$, $s \in e$, and $\chi_e(s) = 0$, $s \notin e$ (where e is a measurable subset of I or $I \times I$). The fundamental function $\phi_E(t) = \|\chi_{(0,t)}\|_E$ of the r.i. space E and the dilation operator $\sigma_t y(u) = y(u/t)\chi_{[0,1]}(u/t)$ $(t > 0)$, which is continuous in any r.i. space, play an important role in the theory of r.i. spaces.

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Theorem 1. *For any r.i. space E on I we have*

$$\phi_{M(E)}(t) = \|\sigma_t\|_{E\to E}, \qquad 0 < t \le 1,$$
$$\|\sigma_t\|_{M(E)\to M(E)} = \|\sigma_t\|_{E\to E}, \qquad 0 < t \le 1, \tag{1}$$
$$\|\sigma_{1/t}\|^{-1}_{E\to E} \le \|\sigma_t\|_{M(E)\to M(E)} \le \|\sigma_t\|_{E\to E}, \qquad t > 1. \tag{2}$$

In general, inequalities (2) cannot be replaced by a relation similar to (1). To establish this it suffices to consider the case in which $\phi_\alpha(s) = s^\alpha \ln^{-1}(C/s)$, $E = \Lambda(\phi_\alpha)$ $(0 < \alpha < 1)$, and $C > \exp\{1/(1-\alpha)\}$ and apply the result of the above example.

Recall that for any r.i. space E, the limits

$$\alpha_E = \lim_{t\to 0} \ln \|\sigma_t\|_{E\to E}/\ln t, \qquad \beta_E = \lim_{t\to\infty} \ln \|\sigma_t\|_{E\to E}/\ln t$$

exist, which are called the Boyd upper and lower indices of the space E [1], respectively. We always have $0 \le \alpha_E \le \beta_E \le 1$ [1, p. 134].

Corollary 1. *For the Boyd indices of r.i. spaces E and $M(E)$ the inequalities $\alpha_E \le \alpha_{M(E)} \le \beta_{M(E)} \le \beta_E$ hold.*

Theorem 2 (upper bound). *Let E be an r.i. space on I and let $p = 1/\alpha_E$, where α_E is the Boyd lower index of E. Then we have $M(E) \subset L_p$, and the embedding constant does not depend on E.*

Corollary 2. *If $\alpha_E = 0$, then $M(E) = L_\infty$.*

If E is an r.i. space on I, then the associated space E' consists of all measurable functions $y = y(t)$ on I such that

$$\|y\|_{E'} = \sup\left\{ \int_0^1 x(t)y(t)\,dt;\ \|x\|_E \le 1 \right\} < \infty.$$

The norm in an r.i. space E is said to be *order semicontinuous* if the monotone convergence $x_n \uparrow x$ almost everywhere $(x_n, x \in E)$ implies the relation $\|x_n\| \to \|x\|$.

Corollary 3. *Let E be an r.i. space with order semicontinuous norm. If $B\colon E \times E \to E(I \times I)$ and $B\colon E' \times E' \to E'(I \times I)$, then $E = L_p$ for some $p \in [1, \infty]$.*

Theorem 3 (lower bound). *For any r.i. space E on I we have*

$$M(E) \supset \Lambda(\psi),$$

where $\psi(t) = \|\sigma_t\|_{E\to E}$ $(0 < t \le 1)$.
Here the embedding constant does not depend on E.

Let us pass to concrete results. Recall that the space L_{pq} $(1 < p < \infty,\ 1 \le q \le \infty)$ consists of all measurable functions $x = x(t)$ on I for which the functional

$$\|x\|_{pq} = \begin{cases} \{\int_0^1 (x^*(t)t^{1/p})^q\,dt/t\}^{1/q} & \text{for } 1 \le q < \infty, \\ \sup_{0<t\le 1}(x^*(t)t^{1/p}) & \text{for } q = \infty, \end{cases}$$

which is equivalent to the norm, is finite.

Theorem 4. *Let an r.i. space E be an interpolation space between L_p and $L_{p,\infty}$ for some $p \in (1, \infty)$ (i.e., if a linear operator is bounded in L_p and $L_{p,\infty}$, then it is continuous in E). Then $M(E) = L_p$.*

Since L_{pq} $(p \le q \le \infty)$ is an interpolation space between L_p and $L_{p,\infty}$ [1, p. 142], we obtain

Corollary 4. *If $1 < p < \infty$, $p \le q \le \infty$, then $M(L_{pq}) = L_p$.*

For the case $q = \infty$, the last assertion was proved in [9].

In conclusion we present an application of the above results in connection with the well-known O'Neil theorem on continuity of the tensor product operator in the spaces L_{pq}.

Theorem 5 [8]. *Let $1 < p < \infty$ and $1 \le q, r, s \le \infty$. The operator B from $L_{pr} \times L_{ps}$ into $L_{pq}(I \times I)$ is continuous if and only if the following conditions hold*:

(1) $\max(s, r) \le q$;

(2) $1/p + 1/q \le 1/s + 1/r$.

Corollary 4 shows that Theorem 5 is exact with respect to the class of all r.i. spaces (and not only L_{pq} spaces) in the case $s = p < r = q$. Namely, L_p is the maximal space among the r.i. spaces E for which B is a bounded operator from $L_{pr} \times E$ into $L_{pr}(I \times I)$.

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Infinite-Dimensional Non-Gaussian Analysis and Generalized Translation Operators

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Gaussian infinite-dimensional analysis [1, 2] has been recently generalized to the case of a non-Gaussian measure, and one of such generalizations is based upon the replacement of orthogonal decompositions with biorthogonal ones [3, 4]. In [5–7], in the model one-dimensional case, it was established that the approach of [3, 4] can be widely generalized if one takes the characters of an L_1-hypergroup instead of the exponents and the generalized translations instead of the ordinary ones. In this note we show that the same procedure can be carried out in the infinite-dimensional setting and obtain results, for the general case, which generalize results of [3, 4]. Note that here a detailed theory (still not constructed) of L_1-hypergroups with a non-locally-compact basis Q is not needed and it suffices to use generalized translation operators T_x $(x \in Q)$ which satisfy rather simple conditions.

1. Let Q be a separable complete metric space of points $x, y, \dots$ and let $C(Q)$ be the linear space of all complex-valued locally bounded (i.e., bounded on every ball in Q) continuous functions on Q. Let a family $T = \{T_x\}_{x \in Q}$ of linear operators on $C(Q)$, the so-called "generalized translation operators," be given and have the following properties: (a) $(T_x f)(y) = (T_y f)(x)$ $(x, y \in Q)$ for an arbitrary function $f \in C(Q)$ (commutativity); (b) there is a point $e \in Q$ (basic identity element) such that $T_e = \mathrm{id}$; (c) for arbitrary $x, y \in Q$, there is a ball $W = W_{x,y} \subset Q$ such that, for an arbitrary function $f \in C(Q)$, the

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