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INTERPOLATION OF OPERATORS IN QUASINORMED GROUPS OF MEASURABLE FUNCTIONS

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Recall that a *quasinorm* on an Abelian group X is a real function $\|\cdot\|_X$ that is defined on X and satisfies the conditions: (a) $\|x\|_X \geq 0$, $\|x\| = 0 \iff x = 0$; (b) $\|-x\|_X = \|x\|_X$; and (c) $\|x + y\|_X \leq C(\|x\|_X + \|y\|_X)$, $C \geq 1$. In this case X is called a *quasinormed group*. As is known, without loss of generality we may assume that $C = 1$ [1, pp. 80–81]. It the present article we study complete quasinormed groups X of Lebesgue measurable real functions on the semiaxis $(0,\infty)$ with the conventional addition. We assume that the quasinorm $\|\cdot\|_X$ satisfies the following two conditions: (1) if $|x| \leq |y|$ (i.e., $|x(s)| \leq |y(s)|$ a.e.) and $y \in X$ then $x \in X$ and $\|x\|_X \leq \|y\|_X$; (2) if functions $|x|$ and $|y|$ are equimeasurable (i.e., $n_{|x|}(\tau) = n_{|y|}(\tau)$, $n_z(\tau) = \mu\{s > 0 : z(s) > \tau\}$) and $y \in X$ then $x \in X$ and $\|x\|_X = \|y\|_X$. Such a quasinormed group will be called a *rearrangement invariant group* (r.i. group). An important example of an r.i. group is the scale $L_p = L_p(0,\infty)$, $0 < p \leq \infty$, defined in a routine fashion. For $0 < p \leq 1$, the quasinorm in L_p is defined by the equality

$$\|x\|_p = \int_0^\infty |x(s)|^p \, ds.$$

Thus, following [2] we can pass to the limit case: $p = 0$. We denote by L_0 the set of all measurable functions $x = x(t)$ for which $\|x\|_0 = \mu(\operatorname{supp} x) < \infty$, where $\operatorname{supp} x = \{t > 0 : \ x(t) \neq 0\}$. Then L_0 becomes an r.i. group continuously embedded into the space S of all measurable a.e. finite functions endowed, as usual, with convergence in measure on the sets of finite measure. In this connection we may consider the Banach pair (L_0, L_∞) of r.i. groups whose role in the class of all r.i. groups is similar to the role of the pair (L_1, L_∞) in the class of rearrangement invariant spaces on $(0,\infty)$ [3].

The main goal of the article is to describe the orbit $\operatorname{Orb}(a; L_0, L_\infty)$ in the pair (L_0, L_∞) for an arbitrary function $a \in L_0 + L_\infty$. We have as a corollary some theorem claiming that all r.i. groups intermediate between L_0 and L_∞ are interpolate with respect to the pair. Moreover, it is shown that, generally speaking, the $\mathcal{K}$-orbit of an element $a \in L_0 + L_\infty$ does not coincide with its orbit. These results testify to essential distinction between interpolation in the pair (L_0, L_∞) and that in the pair (L_1, L_∞) [3, Chapter 2, § 4].

1. Description for Orb $(a; L_0, L_\infty)$. A mapping $T : X \to X$, where X is an r.i. group, is called a *homomorphism* if $T(x+y) = Tx + Ty$ and $T(-x) = -T(x)$ for $x, y \in X$. As usual, a homomorphism is called *bounded* if

$$\|T\|_{X\to X} = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} < \infty.$$

Let (X_0, X_1) be a Banach pair of r.i. groups. We define the *orbit* of an element $a \in X_0 + X_1$ to be the set of all $x \in X_0 + X_1$ representable as $x = Ta$, where T is a bounded homomorphism in X_0 and X_1. Furthermore,

$$\|x\|_{\mathrm{Orb}} = \inf \|T\|_{(X_0,X_1)},$$

where the infimum is taken over all homomorphisms T with $Ta = x$; $\|T\|_{(X_0,X_1)} = \max\limits_{i=0,1} \|T\|_{X_i\to X_i}$.

It is easy to check that the quasinorm $\|\cdot\|_{\mathrm{Orb}}$ makes $\operatorname{Orb}(a; X_0, X_1)$ into an r.i. group.

Henceforth we shall denote by $z^*(t)$ the nonincreasing rearrangement of a measurable function $|z(t)|$ [3, p. 83].

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Theorem 1. *Given an arbitrary $a \in L_0 + L_\infty$, the orbit $\mathrm{Orb}(a; L_0, L_\infty)$ coincides with the set of all $x \in L_0 + L_\infty$ for which there exists $C > 0$ such that*

$$x^*(t) \leq Ca^*(t/C). \tag{1}$$

Moreover, $\|x\|_{\mathrm{Orb}}$ equals the greatest lower bound of all C's for which inequality (1) *is satisfied.*

First we establish one auxiliary claim. Given a pair (X_0, X_1) of r.i. groups, we introduce the functional

$$E(t, x; X_0, X_1) = \inf\{\|x - x_0\|_{X_1};\ \|x_0\|_{X_0} \leq t\}, \quad x \in X_0 + X_1,\ t > 0,$$

that plays an important role in approximation theory [1, Chapter 7].

Lemma 1. *If T is a bounded homomorphism in X_0 and X_1 then for $x \in X_0 + X_1$ the following inequality holds:*

$$E(t, Tx;\ X_0, X_1) \leq \|T\| E(t/\|T\|, x; X_0, X_1),$$

where $\|T\| = \|T\|_{(X_0, X_1)}$.

PROOF. Let $y_0 = Tx_0$ and $x_0 \in X_0$. If $\|T\|\,\|x_0\|_{X_0} \leq t$ then $\|y_0\|_{X_0} \leq t$. Thus,

$$\begin{aligned} E(t, Tx; X_0, X_1) &= \inf\{\|Tx - y_0\|_{X_1}; \|y_0\|_{X_0} \leq t\} \\ &\leq \inf\{\|Tx - Tx_0\|_{X_1}; \|x\|_{X_0} \leq t/\|T\|\} \leq \|T\| E(t/\|T\|, x; X_0, X_1). \end{aligned}$$

In view of $E(t, x;\ L_0, L_\infty) = x^*(t)$ [1, p. 227], we obtain

Corollary. *If T is a bounded homomorphism in L_0 and L_∞ then for $x \in L_0 + L_\infty$ the following inequality holds:*

$$(Tx)^*(t) \leq \|T\| x^*(t/\|T\|), \quad t > 0, \tag{2}$$

where $\|T\| = \|T\|_{(L_0, L_\infty)}$.

PROOF OF THEOREM 1. First assume $x \in \mathrm{Orb}(a; L_0, L_\infty)$. For every $\varepsilon > 0$, there exists a homomorphism T such that $Ta = x$ and $\|T\| = \|T\|_{(L_0, L_\infty)} \leq \|x\|_{\mathrm{Orb}} + \varepsilon$. Applying (2), we obtain the estimate

$$x^*(t) \leq \|T\| a^*(t/\|T\|) \leq (\|x\|_{\mathrm{Orb}} + \varepsilon) a^*(t/(\|x\|_{\mathrm{Orb}} + \varepsilon)).$$

Since $\varepsilon > 0$ is arbitrary and the rearrangement $a^*(s)$ is a.e. continuous, we have

$$x^*(t) \leq \|x\|_{\mathrm{Orb}} a^*(t/\|x\|_{\mathrm{Orb}}). \tag{3}$$

Thus, inequality (1) holds for $C = \|x\|_{\mathrm{Orb}}$.

Conversely, assume that (1) holds for $C = 1$; i.e.,

$$x^*(t) \leq a^*(t), \quad t > 0. \tag{4}$$

Assign $a^*(\infty) = \lim\limits_{t \to \infty} a^*(t)$ and $x^*(\infty) = \lim\limits_{t \to \infty} x^*(t)$, and consider the following two situations: (a) $a^*(\infty) = 0$ and (b) $a^*(\infty) > 0$.

(a) In this case $x^*(\infty) = 0$. Given an arbitrary $\varepsilon > 0$, we have to find a homomorphism T possessing the properties $x = Ta$ and $\|T\|_{(L_0, L_\infty)} \leq 1 + \varepsilon$. We will construct it as a composition of five homomorphisms.

Define first of them: $T_1 y(t) = \operatorname{sign} a(t) y(t)$, $\|T_1\|_{(L_0, L_\infty)} = 1$, $T_1 a = |a|$.

To construct T_2, introduce the sets

$$E_0 = \{t : |a(t)| > 1\}, \quad E_n = \left\{t : \frac{1}{n+1} < |a(t)| \leq \frac{1}{n}\right\}, \ n \in \mathbb{N}.$$

Denote similar sets for a^* by E_n^*. In view of equimeasurability of $|a|$ and a^* and the inequality $a^*(\infty) = 0$, we have

$$\mu(E_n) = \mu(E_n^*) < \infty, \quad \operatorname{supp} a = \bigcup_{n=0}^{\infty} E_n, \quad \operatorname{supp} a^* = \bigcup_{n=0}^{\infty} E_n^*.$$

Consequently, the functions $|a|\mathbf{1}_{E_n}$ and $a^*\mathbf{1}_{E_n^*}$ are equimeasurable for all $n = 0, 1, \dots$. (Here $\mathbf{1}_E$ denotes the indicator of a set E; i.e., $\mathbf{1}_E(s) = 1$ for $s \in E$, and $\mathbf{1}_E(s) = 0$ for $s \notin E$.)

In view of [3, p. 84], for all $\varepsilon > 0$ and $n = 0, 1 \dots$, there exists a measure-preserving mapping $\omega_n : E_n^* \to E_n$ such that

$$\|a^*\mathbf{1}_{E_n^*} - (|a|\mathbf{1}_{E_n})(\omega_n)\|_\infty \le \varepsilon/(n+1). \tag{5}$$

Assign $z_n = a^*\mathbf{1}_{E_n^*} - (|a|\mathbf{1}_{E_n})(\omega_n)$ and $\alpha_n(t) = z_n(t)/(|a|\mathbf{1}_{E_n})(\omega_n(t))$ for $t \in E_n^*$ and $\alpha_n(t) = 0$ for $t \notin E_n^*$. Then (5) and the definition of E_n imply that

$$\|\alpha_n\|_\infty \le \varepsilon, \quad n = 0, 1, 2, \dots. \tag{6}$$

Moreover, $a^*\mathbf{1}_{E_n^*} = |a|\mathbf{1}_{E_n}(\omega_n) + z_n = (1 + \alpha_n)(|a|\mathbf{1}_{E_n})(\omega_n)$ and, after summing over all n, we obtain $a^*(t) = (1 + \alpha(t))|a(\omega(t))|$, where the mappings ω and α result from "fusion" of ω_n and α_n ($n = 0, 1, \dots$) respectively (for $t \notin \operatorname{supp} a^*$ we put $\alpha(t) = 0$ and $\omega(t) = s_0$, where $s_0 \notin \operatorname{supp} a$). Then the mapping $\omega : (0, \infty) \to (0, \infty)$ does not increase the measure of sets and $\|\alpha\|_\infty \le \varepsilon$ by (6). Define the homomorphism $T_2 y(t) = (1 + \alpha(t))y(\omega(t))$. We have $\|T_2\|_{(L_0, L_\infty)} \le 1 + \varepsilon$ and $T_2|a| = a^*$. Further, put $T_3 y(t) = \beta(t)y(t)$, where $\beta(t) = x^*(t)/a^*(t)$ if $a^*(t) > 0$ and $\beta(t) = 0$ if $a^*(t) = 0$. By (4), $|\beta(t)| \le 1$. Hence, $\|T_3\|_{(L_0, L_\infty)} \le 1$ and $T_3 a^* = x^*$.

Construct the homomorphism T_4 by analogy with T_2 but, instead of E_0 and E_n, consider the sets

$$F_0^* = \{t : x^*(t) > 1\}, \quad F_n^* = \left\{t : \frac{1}{n+1} < x^*(t) \le \frac{1}{n}\right\}$$

and, instead of E_n^* and E_n^*, the set

$$F_0 = \{t : |x(t)| > 1\}, \quad F_n = \left\{t : \frac{1}{n+1} < |x(t)| \le \frac{1}{n}\right\}.$$

Then $\|T_4\|_{(L_0, L_\infty)} \le 1 + \varepsilon$ and $T_4 x^* = |x|$.

Finally, $T_5 y(t) = \operatorname{sign} x(t) y(t)$, $\|T_5\|_{(L_0, L_\infty)} = 1$, $T_5|x| = x$.

The homomorphism $T = T_5 T_4 T_3 T_2 T_1$ is bounded in L_0 and L_∞,

$$\|T\|_{(L_0, L_\infty)} \le (1 + \varepsilon)^2, \quad Ta = x.$$

(b) Assume $a^*(\infty) = \delta > 0$. Define the homomorphism T_1, $T_1 a = |a|$, as in case (a).

By using [3, p. 84], find again a measure-preserving mapping $\omega = \omega(t)$ of the semiaxis into itself satisfying the inequality

$$\|a^* - |a(\omega)|\,\|_\infty \le \varepsilon\delta/2, \quad 0 < \varepsilon < 1.$$

Since $a^*(t) \ge \delta$, we have $|a(\omega(t))| \ge \delta/2$ a.e. Consequently, by denoting $z = a^* - |a(\omega)|$ and $\alpha(t) = z(t)/(|a(\omega(t))|)$ if $a(\omega(t)) \ne 0$ and $\alpha(t) = 0$ if $a(\omega(t)) = 0$, we obtain

$$\|\alpha\|_\infty \le \varepsilon. \tag{7}$$

Thus, $a^* = |a(\omega)| + z = (1 + \alpha)|a(\omega)|$. Define the homomorphism $T_2 y(t) = (1 + \alpha(t))y(\omega(t))$. By (7), $\|T_2\|_{(L_0, L_\infty)} \le 1 + \varepsilon$ and $T_2|a| = a^*$.

Define the homomorphism T_3, $T_3 a^* = x^*$, as in case (a).

If $x^*(\infty) = 0$ then T_4 is the same as in the previous case. If $x^*(\infty) = \delta_1 > 0$ then, for $x_1(t) = \max(|x(t)|, \delta_1)$, find a measure-preserving mapping $\omega_1 : (0,\infty) \to (0,\infty)$ such that

$$\left\|x_1 - x^*(\omega_1)\right\|_\infty \leq \varepsilon\delta_1/2.$$

Then $x^*(\omega_1(t)) \geq \delta_1/2$ and, by putting $z_1 = x_1 - x^*(\omega_1)$ and $\alpha_1(t) = z_1(t)/x^*(\omega_1(t))$ if $x^*(\omega_1(t)) \neq 0$ and $\alpha_1(t) = 0$ if $x^*(\omega_1(t)) = 0$, obtain $\|\alpha_1\|_\infty \leq \varepsilon$. Consequently, the homomorphism T_4, $T_4y(t) = (1+\alpha_1(t))y(\omega_1(t))$, is bounded in L_0 and L_∞, $\|T_4\|_{(L_0,L_\infty)} \leq 1+\varepsilon$, $T_4x^* = x_1$. Finally, $T_5y(t) = \beta_1(t)y(t)$, where $\beta_1(t) = x(t)/x_1(t)$ if $x_1(t) \neq 0$ and $\beta_1(t) = 0$ if $x_1(t) = 0$. Since $|\beta_1(t)| \leq 1$, we have $\|T_5\|_{(L_0,L_\infty)} \leq 1$ and $T_5x_1 = x$.

By putting $T = T_5T_4T_3T_2T_1$, we obtain $\|T\|_{(L_0,L_\infty)} \leq (1+\varepsilon)^2$ and $Ta = x$ as in case (a).

Now, let inequality (1) be satisfied for some $C > 0$. Assign $y(t) = x(t)/C$ and $b(t) = a(t/C)$. Then $y^*(t) \leq b^*(t)$ and, by what was proven, for $\varepsilon > 0$ there exists a homomorphism T with

$$\|T\|_{(L_0,L_\infty)} \leq (1+\varepsilon)^2, \quad Tb = y.$$

Define the homomorphisms $R_Cz(t) = z(t/C)$ and $Q_Cz(t) = Cz(t)$. Since $\|R_C\|_{L_\infty\to L_\infty} = \|Q_C\|_{L_0\to L_0} = 1$ and $\|Q_C\|_{L_\infty\to L_\infty} = \|R_C\|_{L_0\to L_0} = C$, for $V = Q_CTR_C$ we have $\|V\|_{(L_0,L_\infty)} \leq (1+\varepsilon)^2C$ and $Va = x$.

Since $\varepsilon > 0$ is arbitrary, it now follows from (1) that $x \in \mathrm{Orb}(a; L_0, L_\infty)$ and $\|x\|_{\mathrm{Orb}} \leq C$. Together with inequality (3) it means that Theorem 1 is proven.

2. Interpolation in the pair (L_0, L_∞). Recall that an r.i. group X is called *interpolate* between r.i. groups X_0 and X_1 if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ (with all embeddings continuous) and every homomorphism bounded in X_0 and X_1 is bounded in X.

We exhibit two auxiliary results.

Lemma 2. *The following assertions are valid for every r.i. group X on $(0,\infty)$:*

(1) if $x \in X$ and $C > 0$ then $Cx \in X$ and $\|Cx\|_X \leq ([C]+1)\|x\|_X$ ($[C]$ is the integral part of C);

(2) for each $\tau > 0$, the dilatation homomorphism $R_\tau x(t) = x(t/\tau)$ is bounded in X and $\|R_\tau\|_{X\to X} \leq [\tau]+1$.

PROOF. The first assertion of the lemma is an immediate consequence of the two properties of a quasinorm, monotonicity and subadditivity.

Given a natural τ, as in the case of rearrangement invariant spaces [3, pp. 131–132] one can demonstrate that $\|R_\tau\|_{X\to X} \leq \tau$. For the other τ's, the second assertion follows from the relations

$$(R_{\tau_1}x)^*(t) = \left(R_{\tau_1}x^*\right)(t) \leq \left(R_{\tau_2}x^*\right)(t) = (R_{\tau_2}x)^*(t) \quad (\tau_1 < \tau_2).$$

Lemma 3. *Let X be an r.i. group with quasinorm $\|\cdot\|$. If $\sum_{n=1}^{\infty}\|x_n\| < \infty$ for $x_n \in X$, then the function series $\sum_{n=1}^{\infty}|x_n(t)|$ converges a.e.*

PROOF. Assume that the first assertion of the lemma is false; i.e., there exists an E, $0 < \mu(E) < \infty$, such that $z(t) = \sum_{n=1}^{\infty}|x_n(t)| = \infty$ for $t \in E$. Denote $E_n^m = \left\{t \in E : \sum_{i=n}^{m}|x_i(t)| \geq 1\right\}$, $n \leq m$. Given an arbitrary $n \in \mathbb{N}$, we have $E_n^m \subset E_n^{m+1}$ and $E = \bigcup_{m=n}^{\infty} E_n^m$. Consequently, $\mu(E) = \lim_{m\to\infty}\mu(E_n^m)$. Choose m_n so as to have $\mu\left(E \setminus E_n^{m_n}\right) < \mu(E)/2^{n+1}$. Then $E_0 = \bigcap_{n=1}^{\infty} E_n^{m_n} \subset E$ and

$$\mu(E_0) = \mu(E) - \mu(E \setminus E_0) = \mu(E) - \mu\left(\bigcup_{n=1}^{\infty}\left(E \setminus E_n^{m_n}\right)\right) > \mu(E)/2 > 0.$$

On the other hand, we have

$$\|\mathbf{1}_{E_0}\| \le \sum_{i=n}^{\infty} \|x_i\|, \quad n = 1, 2, \dots,$$

since $\mathbf{1}_{E_0} \le \sum_{i=n}^{m_n} |x_i|$ by the definition of E_0. Thus, $\|\mathbf{1}_{E_0}\| = 0$; i.e., $\mu(E_0) = 0$. The contradiction obtained proves the lemma.

Assume that the quasinorm $\|\cdot\|_X$ on an r.i. group X possesses the property

$$\inf\{\|x\|_X; x \ne 0\} > 0. \tag{8}$$

Then it generates discrete topology in X. We shall call a quasinorm with property (8) *discrete.*

Theorem 2. *Let X be a nonempty r.i. group with nondiscrete quasinorm and let $X \ne S$, where S is the set of all measurable a.e. finite functions on the semiaxis. Then $L_0 \cap L_\infty \subset X \subset L_0 + L_\infty$; moreover, the embeddings are continuous.*

PROOF. Assume that $x \in L_0 \cap L_\infty$; i.e., $\alpha = \mu(\operatorname{supp} x) < \infty$ and $\|x\|_\infty < \infty$ (we may suppose that $\alpha > 0$). Since X is nonempty, there exists an $x_0 \in X$, $x_0 \ne 0$. Furthermore, there exist a measurable set E and an $\varepsilon > 0$ such that $0 < \mu(E) = \beta < \|x\|_{L_0 \cap L_\infty}$, $\varepsilon < \|x\|_{L_0 \cap L_\infty}$, and $z = \varepsilon \mathbf{1}_E \le |x_0|$. Thereby, $z \in X$ and $\|z\|_X \le \|x_0\|_X$.

Define the function $w(t) = \|x\|_\infty \mathbf{1}_E((\beta/\alpha)t)$. By Lemma 2, $w \in X$ and

$$\|w\|_X \le ([\alpha/\beta] + 1)([\|x\|_\infty/\varepsilon] + 1)\|z\|_X \le (4/\varepsilon\beta)\|x_0\|_X \|x\|_{L_0 \cap L_\infty}$$

(ε and β are independent of x). Since $x^* \le w^*$, we obtain $x \in X$.

Continuity of the embedding $L_0 \cap L_\infty \subset X$ follows from the fact that the norm $\|\cdot\|_X$ is not discrete. Indeed, given an arbitrary $\delta > 0$, find $x \in X$ such that $x \ne 0$ and $\|x\|_X < \delta$. As above, there exist a $\gamma > 0$ and a measurable set F, $\mu(F) > 0$, for which $z = \gamma \mathbf{1}_F \le |x|$. Thus, $\|z\|_X < \delta$. If now $\|x_n\|_{L_0 \cap L_\infty} \to 0$ and $\|x_n\|_{L_0 \cap L_\infty} \le \min(\gamma, \mu(F))$ for $n \ge N$, then $x_n^* \le z^*$, whence $\|x_n\|_X = \|x_n^*\|_X \le \|z\|_X < \delta$. Consequently, $\|x_n\|_X \to 0$.

Demonstrate that $X \not\subset L_0 + L_\infty$ if and only if $X = S$. Indeed, assume that there exists an $x_0 \in X$, $x_0 \notin L_0 + L_\infty$. The last means that the distribution function $n_{|x_0|}(\tau)$ is identically infinite. Since X is an r.i. group, all functions $z = z(t)$ for which $n_{|z|}(\tau) \equiv \infty$, $\tau > 0$, belong to X. If now $x(t)$ is arbitrary then $|x| \le z = \max(|x_0|, |x|)$ and $n_{|z|}(\tau) \equiv \infty$. Thus, $x \in X$.

In conclusion we prove that the embedding $X \subset L_0 + L_\infty$ is continuous. Assume to the contrary that there is a sequence $\{x_n\} \subset X$ such that $\|x_n\|_X \to 0$ whereas

$$\|x_n\|_{L_0 + L_\infty} \ge \sigma > 0, \quad n = 1, 2, \dots. \tag{9}$$

Find a subsequence $\{\bar{x}_n\} \subset \{x_n\}$ for which

$$\sum_{n=1}^{\infty} \|\bar{x}_n^*\|_X < \infty.$$

By Lemma 3, the series $\sum_{n=1}^{\infty} \bar{x}_n^*(t)$ converges a.e. on $(0, \infty)$ and hence $\bar{x}_n^*(t) \to 0$ a.e. Consequently, by Riezs's theorem, there exists a subsequence $\{\bar{\bar{x}}_n\} \subset \{\bar{x}_n\}$ such that $\mu\{t \in (0,1] : \bar{\bar{x}}_n^*(t) > \tau\} \to 0$ for $\tau > 0$. Since the functions $\bar{\bar{x}}_n^*$ do not increase; therefore, $\bar{\bar{x}}_n^* \to 0$ in measure on the whole semiaxis.

As is known [1, p. 223], we have

$$\|x\|_{L_0 + L_\infty} = \inf\{t + x^*(t); t > 0\}.$$

Straightforward verification demonstrates that the quasinorm determines convergence in measure on the semiaxis. Consequently, by what was proven above, we have

$$\|\bar{\bar{x}}_n\|_{L_0 + L_\infty} = \|\bar{\bar{x}}_n^*\|_{L_0 + L_\infty} \to 0,$$

which contradicts (9). The theorem is proven.

Theorem 3. *Every nonempty r.i. group X, $X \neq S$, with nondiscrete quasinorm interpolates between the groups L_0 and L_∞; i.e., every homomorphism bounded in L_0 and L_∞ is bounded in X.*

PROOF. By Theorem 2, each homomorphism defined on L_0 and L_∞ is defined on X and inequality (2) holds for all $x \in X$. Hence, by Lemma 2,

$$\|Tx\|_X \leq ([\|T\|] + 1)^2 \|x\|_X.$$

REMARK 1. Theorem 1 readily yields some assertion that is in some sense converse of the claim of Theorem 3: If a quasinormed group of measurable functions interpolates between L_0 and L_∞ then it is rearrangement invariant.

REMARK 2. As is known, the situation is different with rearrangement invariant spaces: among them there are spaces intermediate between L_1 and L_∞ but not interpolate with respect to this pair [3, pp. 166–169].

3. On the $\mathcal{K}$-orbit in the pair (L_0, L_∞). Let (X_0, X_1) be a Banach pair of r.i. groups and $a \in X_0 + X_1$. As in the case of Banach spaces [4, p. 407], we define the $\mathcal{K}$-*orbit* of a to be the set $\mathcal{KO}(a; X_0, X_1)$ of all $x \in X_0 + X_1$ for which

$$\|x\|_{\mathcal{KO}} = \sup_{t>0} \frac{\mathcal{K}(t, x; X_0, X_1)}{\mathcal{K}(t, a; X_0, X_1)} < \infty,$$

where

$$\mathcal{K}(t, z;\ X_0, X_1) = \inf\{\|z_0\|_{X_0} + t\|z_1\|_{X_1};\ z = z_0 + z_1, z_i \in X_i\}.$$

A pair (X_0, X_1) is called $\mathcal{K}$-*monotone* if $\mathcal{KO}(a; X_0, X_1) = \operatorname{Orb}(a; X_0, X_1)$ for all $a \in X_0 + X_1$. The pair (L_1, L_∞) is a classical example of a $\mathcal{K}$-monotone pair [5, 6] (for other examples of $\mathcal{K}$-monotone pairs see [4]).

Denote by 1-$\mathcal{KO}(a; X_0, X_1)$ and 1- $\operatorname{Orb}(a; X_0, X_1)$ the closed unit balls in r.i. groups $\mathcal{KO}(a; X_0, X_1)$ and $\operatorname{Orb}(a; X_1, X_1)$. It is easy to check that

$$1\text{-}\operatorname{Orb}(a; X_0, X_1) \subset 1\text{-}\mathcal{KO}(a; X_0, X_1).$$

We now show that the reverse inclusion may fail for the pair $(X_0, X_1) = (L_0, L_\infty)$.

Theorem 4. *There exist $a \in L_0 + L_\infty$ and $x \in L_0 + L_\infty$ such that*

$$\mathcal{K}(t, a;\ L_0, L_\infty) \equiv \mathcal{K}(t, x;\ L_0, L_\infty),$$

whereas $a^*(t) < x^*(t)$, $t \in E$, for some measurable set E, $\mu(E) > 0$.

PROOF. Given numbers $0 < x_1 < x_0$ and $0 < \tau_1 < \tau_2$, suppose that the inequality

$$x_1\tau_1 < (\tau_2 - \tau_1)(x_0 - x_1) \tag{10}$$

holds. Define the step function

$$x(t) = x_0 \mathbf{1}_{(0,\tau_1]}(t) + x_1 \mathbf{1}_{(\tau_1,\tau_2)}(t).$$

Since $\mathcal{K}(t, y; L_0, L_\infty) = \inf\{s + ty^*(s); s > 0\}$, $y \in L_0 + L_\infty$ [1, p. 223], we have

$$\mathcal{K}(t, x;\ L_0, L_\infty) = \min\{tx_0, tx_1 + \tau_1, \tau_2\}.$$

It is straightforward from (10) that

$$\mathcal{K}(t, x;\ L_0, L_\infty) = \begin{cases} tx_0, & t \leq \tau_1/(x_0 - x_1), \\ tx_1 + \tau_1, & \tau_1/(x_0 - x_1) < t \leq (\tau_2 - \tau_1)/x_1, \\ \tau_2, & t > (\tau_2 - \tau_1)/x_1. \end{cases} \tag{11}$$

Now we diminish $x(t)$ on the interval (τ_1, τ_2) while making it piecewise linear and continuous and demonstrate that the $\mathcal{K}$-functional remains nevertheless invariable. More precisely, we define the function

$$a(t) = x_0 \mathbf{1}_{(0,\tau_1]}(t) + (x_1 - h(\tau_1 + t))\mathbf{1}_{(\tau_1,\tau_2)}(t),$$

where $h(\tau_1 + s) = (sx_1/(\tau_2 - \tau_1))\mathbf{1}_{[0,\tau_2-\tau_1]}(s)$, and prove that

$$\mathcal{K}(t, a;\, L_0, L_\infty) \equiv \mathcal{K}(t, x;\, L_0, L_\infty), \quad t > 0. \tag{12}$$

Consider the following three cases:

1. $t \le \tau_1/(x_0 - x_1)$. It suffices to establish validity for the inequality

$$tx_0 \le \tau_1 + s + t(x_1 - h(\tau_1 + s)), \quad 0 < s < \tau_2 - \tau_1,$$

which amounts to

$$h(\tau_1 + s) \le x_1 - x_0 + (\tau_1 + s)/t.$$

It suffices to test the preceding inequality for $t = \tau_1/(x_0 - x_1)$:

$$h(\tau_1 + s) \le s(x_0 - x_1)/\tau_1,$$

with the last inequality following from the definition of the function $h(\tau_1 + s)$ and condition (10).

2. $\tau_1/(x_0 - x_1) < t \le (\tau_2 - \tau_1)/x_1$. We have to verify validity for the inequality

$$h(\tau_1 + s) \le s/t, \quad 0 < s < \tau_2 - \tau_1,$$

which obviously holds on the interval under consideration.

3. $t > (\tau_2 - \tau_1)/x_1$. We have to show that

$$\tau_2 \le \tau_1 + s + t(x_1 - h(\tau_1 + s)), \quad 0 < s < \tau_2 - \tau_1,$$

or

$$h(\tau_1 + s) \le x_1 + (s + \tau_1 - \tau_2)/t.$$

The second summand on the right-hand side is negative; thus, it suffices to take $t = (\tau_2 - \tau_1)/x_1$ and again use the definition of the function $h(\tau_1 + s)$.

Identity (12) is proven. At the same time, we have $x^*(t) = x(t) > a(t) = a^*(t)$ for $\tau_1 < t < \tau_2$. The theorem is proven.

From Theorems 1 and 4 we obtain

Corollary. *There exists a function $a \in L_0 + L_\infty$ for which*

$$1\text{-}\mathrm{Orb}(a;\, L_0, L_\infty) \ne 1\text{-}\mathcal{KO}(a;\, L_0, L_\infty).$$

REMARK 3. Slightly modifying the proof of Theorem 4 (by considering countably-valued step functions), one can demonstrate that there exists a function $a \in L_0 + L_\infty$ for which

$$\mathrm{Orb}(a;\, L_0, L_\infty) \ne \mathcal{KO}(a;\, L_0, L_\infty).$$

Hence, the pair (L_0, L_∞), is not $\mathcal{K}$-monotone as opposed to the pair (L_1, L_∞) (see [3]).

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