# ON CONES OF STEP FUNCTIONS IN REARRANGEMENT INVARIANT SPACES

**S. V. Astashkin**

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**1.** We recall that a *rearrangement invariant space* (r.i. space) is a Banach space $X$ of measurable functions on $[0,1]$ for which

(1) from $y \in X$ and $|x(t)| \leq |y(t)|$ it follows that $x \in X$ and $\|x\|_X \leq \|y\|_X$;

(2) if $y \in X$ and $|x(t)|$ is equimeasurable with $|y(t)|$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

The theory of r.i. spaces is presented in [1].

Let $X$ be an r.i. space on $[0,1]$. Then it is uniquely determined by the cone of functions $y \in X$ of the form

$$y(t) = \sum_{k=1}^{\infty} c_k \chi_{(2^{-k}, 2^{-k+1}]}(t),$$

where $c_k \geq 0$, $c_k \leq c_{k+1}$ $(k = 1, 2, \dots)$, and $\chi_e$ denotes the characteristic function of a set $e \subset [0,1]$. Indeed, given $x \in X$, put

$$y(t) = \sum_{k=1}^{\infty} x^*(2^{-k+1}) \chi_{(2^{-k}, 2^{-k+1}]}(t),$$

where $x^*(t)$ is the nondecreasing rearrangement of $|x(t)|$ [1]. Then $y \leq x^* \leq \sigma_2 y$, where $\sigma_s z(u) = z(u/s)\chi_{[0,1]}(u/s)$ is the dilation operator. Since $\|\sigma_s\|_{X \to X} \leq \max(1, s)$ [1], the preceding inequality implies that $\|y\|_X \leq \|x\|_X \leq 2\|y\|_X$.

Let $\Phi$ be a class of r.i. spaces, and let $X_0 \in \Phi$. We say that a system $\{e_k\}_{k=1}^{\infty}$ of subsets of $[0,1]$ *determines* $X_0$ *in the class* $\Phi$ if the equivalence of the norms of $X_0$ and of an arbitrary r.i. space $X \in \Phi$ on the cone of functions of the form

$$y(t) = \sum_{k=1}^{\infty} c_k \chi_{e_k}(t) \quad (c_k \geq 0, c_k \leq c_{k+1},\ k = 1, 2, \dots) \tag{1}$$

implies that $X = X_0$ (isomorphism).

In the case when $\Phi$ is the class of all maximal r.i. spaces [1], the problem was considered in [2]. The answer obtained there testifies to some degeneracy of the problem as stated: if, for instance, the fundamental function of $X_0$ is polynomial, then a system $\{e_k\}$ of pairwise disjoint subsets determines $X_0$ if and only if $\sup_{k=1,2,\dots} a_k/a_{k+1} < \infty$ for the measures $\mu(e_k) = a_k$ (as a matter of fact, we arrive at the case investigated above: $a_k = 2^{-k}$ ). The reason behind that lies in "too" strong influence of the fundamental function of the r.i. space in question. Seemingly, the problem becomes more relevant if we consider the classes consisting of r.i. spaces with the same fundamental function.

Now, we recall some definitions. Besides $L_p$ $(1 \leq p \leq \infty)$, among important examples of r.i. spaces are the Lorentz space $\Lambda(\varphi)$ and the Marcinkiewicz space $\mathrm{M}(\varphi)$:

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(s)\, d\varphi(s), \quad \|x\|_{\mathrm{M}(\varphi)} = \sup_{0<t\leq 1} \frac{\int_0^t x^*(s)\, ds}{\varphi(t)},$$

where $\varphi(t)$ is an increasing concave function on $[0,1]$, $\varphi(0) = 0$. The equality

$$\mathcal{M}_{\varphi}(t) = \sup_{0<s\leq \min(1,1/t)} \frac{\varphi(st)}{\varphi(s)} \quad (t > 0)$$

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determines the dilation function for $\varphi(t)$. The behavior of $\mathcal{M}_\varphi(t)$ characterizes the lower and upper dilation exponents

$$\gamma_\varphi = \lim_{t\to 0+} \frac{\ln \mathcal{M}_\varphi(t)}{\ln t}, \quad \delta_\varphi = \lim_{t\to\infty} \frac{\ln \mathcal{M}_\varphi(t)}{\ln t}. \tag{2}$$

In particular, if we take as $\varphi(t)$ the fundamental function $\varphi_X(t) = \|\chi_{(0,t)}\|_X$ of an r.i. space $X$ (for $L_p$, it equals $t^{1/p}$; for $\Lambda(\varphi)$, $\varphi(t)$; and for $\mathrm{M}(\varphi)$, $\widetilde{\varphi}(t) = t/\varphi(t)$), then we obtain the numbers $\gamma_X = \gamma_{\varphi_X}$ and $\delta_X = \delta_{\varphi_X}$. If we substitute the norm $\|\sigma_t\|_{X\to X}$, of the dilation operator, for $\mathcal{M}_\varphi(t)$ in (2), then we arrive at the definition of the Boyd indices $\alpha_X$ and $\beta_X$ of the space $X$.

**2.** In what follows we consider systems $\{e_k\}_{k=1}^\infty$ of subsets of $[0,1]$ whose measures $\mu(e_k) = a_k$ satisfy the relations

$$0 < a_{k+1} \le a_k, \quad \sum_{k=1}^\infty a_k \le 1, \tag{3}$$

$$\sup_{k=1,2,\dots} \frac{1}{a_k} \sum_{n=k}^\infty a_n < \infty. \tag{4}$$

As is well known, condition (4) means that the sequence $\{a_k\}$ can be partitioned into a finite number of subsequences so that the members of each subsequence decrease not slower than a geometric progression [3, p. 22].

We recall that a space $X$ is referred to as an *interpolation* space between Banach spaces $X_0$ and $X_1$ if every linear operator bounded over $X_0$ and $X_1$ is bounded over $X$.

**Theorem 1.** *Let $\psi(t)$ be an increasing concave function on $[0,1]$, $0 < \gamma_\psi \le \sigma_\psi < 1$, and let an r.i. space $X$ on $[0,1]$ be an interpolation space between $\Lambda(\psi)$ and $\mathrm{M}(\widetilde{\psi})$, $\overline{\chi}_b = \chi_b/\|\chi_b\|_X$ $(b \subset [0,1])$. If $\Delta_1 = \{e_k\}_{k=1}^\infty$ and $\Delta_2 = \{f_k\}_{k=1}^\infty$ are two systems of subsets of $[0,1]$ for which conditions (3) and (4) hold, then*

$$\left\|\sum_{k=1}^\infty c_k \overline{\chi}_{e_k}\right\|_X \approx \left\|\sum_{k=1}^\infty c_k \overline{\chi}_{f_k}\right\|_X$$

*for arbitrary numbers $c_k \ge 0$ $(k = 1, 2, \dots)$; i.e., there exists an $A > 0$ independent of $\{c_k\}$ and such that*

$$A^{-1}\left\|\sum_{k=1}^\infty c_k \overline{\chi}_{e_k}\right\|_X \le \left\|\sum_{k=1}^\infty c_k \overline{\chi}_{f_k}\right\|_X \le A\left\|\sum_{k=1}^\infty c_k \overline{\chi}_{e_k}\right\|_X.$$

PROOF. Given a system of pairwise disjoint sets $\Delta = \{g_k\}_{k=1}^\infty$, $g_k \subset [0,1]$, put $T_\Delta$ to denote the averaging operator

$$T_\Delta x(t) = \sum_{i=1}^\infty \frac{1}{\mu(g_i)} \int_{g_i} x(s)\,ds\ \chi_{g_i}(t).$$

Since $\|T_\Delta\|_{L_\infty\to L_\infty} = \|T_\Delta\|_{L_1\to L_1} = 1$ and since $\Lambda(\psi)$ and $\mathrm{M}(\widetilde{\psi})$ are exact interpolation spaces between $L_1$ and $L_\infty$ [1, § 5], we have

$$\|T_\Delta\|_{\Lambda(\psi)\to\Lambda(\psi)} \le 1, \quad \|T_\Delta\|_{\mathrm{M}(\widetilde{\psi})\to\mathrm{M}(\widetilde{\psi})} \le 1.$$

Therefore, $T_\Delta$ is bounded in $X$.

From the conditions of the theorem it follows that the value $\|\sum_{k=1}^\infty c_k\overline{\chi}_{e_k}\|_X$ depends only on the numbers $c_k \ge 0$ and $\mu(e_k)$ [4]. Therefore, without loss of generality, we can assume that the sets $e_k$ (and, similarly, $f_k$) are pairwise disjoint.

On the image of $T_{\Delta_1}(\mathrm{M}(\widetilde{\psi}))$, we define the operator $S$:

$$S\left(\sum d_k \overline{\chi}_{e_k}\right) = \sum d_k \overline{\chi}_{f_k}.$$

As was shown in [4], if $0 < \gamma_\psi \leq \delta_\psi < 1$ then

$$\left\|\sum d_k \overline{\chi}_{e_k}\right\|_{\Lambda(\psi)} \approx \left\|\sum d_k \overline{\chi}_{f_k}\right\|_{\Lambda(\psi)} \approx \sum_k |d_k|, \quad \left\|\sum d_k \overline{\chi}_{e_k}\right\|_{\mathrm{M}(\widetilde{\psi})} \approx \left\|\sum d_k \overline{\chi}_{f_k}\right\|_{\mathrm{M}(\widetilde{\psi})} \approx \sup_k |d_k| \quad (5)$$

for all $d_k$ $(k = 1, 2, \ldots)$. Therefore, the operator $ST_{\Delta_1}$ is continuous over $X$ and

$$\left\|\sum c_k \overline{\chi}_{f_k}\right\|_X = \left\|ST_{\Delta_1}\left(\sum c_k \overline{\chi}_{e_k}\right)\right\|_X \leq A_1 \left\|\sum c_k \overline{\chi}_{e_k}\right\|_X,$$

where $A_1 > 0$ is independent of $c_k$, $k = 1, 2, \ldots$. By interchanging $\Delta_1$ and $\Delta_2$, we similarly obtain

$$\left\|\sum c_k \overline{\chi}_{e_k}\right\|_X \leq A_2 \left\|\sum c_k \overline{\chi}_{f_k}\right\|_X$$

for some $A_2 > 0$.

Denote by $I(\psi)$ the class of all r.i. spaces that are interpolation spaces between $\Lambda(\psi)$ and $\mathrm{M}(\widetilde{\psi})$.

**Theorem 2.** *Let a system $\{e_k\}_{k=1}^\infty$, $e_k \subset [0,1]$, satisfy conditions (3) and (4), $X_0 \in I(\psi)$, and $0 < \gamma_\psi \leq \delta_\psi < 1$. Then the system $\{e_k\}$ determines $X_0$ with respect to $I(\psi)$; i.e., the equivalence*

$$\left\|\sum c_k \chi_{e_k}\right\|_X \approx \left\|\sum c_k \chi_{e_k}\right\|_{X_0} \quad (0 \leq c_k \leq c_{k+1},\ k = 1, 2, \ldots)$$

*for $X \in I(\psi)$ implies that $X = X_0$.*

The proof follows from Theorem 1 with $f_k = (2^{-k}, 2^{-k+1}]$ $(k = 1, 2, \ldots)$ and from the remarks made at the beginning of the article.

**Corollary.** *Let an r.i. space $X$ on $[0,1]$ be an interpolation space between $\Lambda(t^{1/p})$ and $\mathrm{M}(t^{1-1/p})$, $1 < p < \infty$. If there exists a system $\{e_k\}_{k=1}^\infty$, $e_k \subset [0,1]$, satisfying (3) and (4) and such that*

$$\left\|\sum_{k=1}^\infty c_k \chi_{e_k}\right\|_X \approx \left(\sum_{k=1}^\infty c_k^p \mu(e_k)\right)^{1/p}$$

*for all $c_k$, $0 \leq c_k \leq c_{k+1}$ $(k = 1, 2, \ldots)$, then $X = L_p$.*

The proof follows from the fact that $L_p$ is an interpolation space between $\Lambda(t^{1/p})$ and $\mathrm{M}(t^{1-1/p})$ [5, p. 148].

A similar result is valid for the general Orlicz spaces [6] as well.

**3.** We will show that assertions like Theorems 1 and 2 of the previous section fail if we "slightly" widen the class of r.i. spaces.

Given an increasing concave function $\psi$ on $[0,1]$, we denote by $J(\psi)$ the class of all r.i. spaces $X$ whose fundamental function is equivalent to $\psi(t)$ and the Boyd indices coincide with the corresponding dilation exponents: $\alpha_X = \gamma_\psi$, $\beta_X = \delta_\psi$. It is easy to show that $I(\psi) \subset J(\psi)$; moreover, the embedding is strict (this, in particular, follows from Theorem 3).

We recall the concept of the real interpolation method.

Let $(X_0, X_1)$ be a Banach pair and let $E$ be an ideal structure of number sequences $a = (a_j)_{j=-\infty}^\infty$ interpolatory between $l_\infty$ and $l_\infty(2^{-k})$ (if $F$ is an ideal structure, $v = (v_k)$, with $v_k \geq 0$, then

$\|(a_j)\|_{F(v_k)} = \|(a_j v_j)\|_F$. The $\mathcal{K}$-method space $(X_0, X_1)^{\mathcal{K}}_E$ (which is an interpolation space between $X_0$ and $X_1$) consists of $x \in X_0 + X_1$ with finite norm

$$\|x\| = \|(\mathcal{K}(2^k, x; X_0, X_1))\|_E,$$

where

$$\mathcal{K}(t, x; X_0, X_1) = \inf_{\substack{x = x_0 + x_1, \\ x_i \in X_i}} \{\|x_0\|_{X_0} + t\|x_1\|_{X_1}\}.$$

Henceforth it will be convenient, for the time being, to deal with rearrangement invariant spaces of functions defined on the half-axes $(0, \infty)$. From the equality

$$\mathcal{K}(t, x; L_1, L_\infty) = \int_0^t x^*(s)\, ds \quad (t > 0)$$

it follows that, for $0 < \theta < 1$, $1 \le p \le \infty$, and $l_{\theta,p} = l_p(2^{k(\theta-1)})$,

$$\Lambda(t^\theta) = (L_1, L_\infty)^{\mathcal{K}}_{l_{\theta,1}}, \tag{6}$$

$$\mathrm{M}(t^{1-\theta}) = (L_1, L_\infty)^{\mathcal{K}}_{l_{\theta,\infty}}, \tag{7}$$

where the Lorentz and Marcinkiewicz spaces on $(0, \infty)$ are defined by analogy with those on $[0, 1]$ [1].

First, we prove an auxiliary claim of independent interest.

**Proposition.** *Let $X$ be an r.i. space on $(0, \infty)$, $X = (L_1, L_\infty)^{\mathcal{K}}_E$, and let $l_\infty \not\subset E$. Then*

$$\|\sigma_{2^k}\|_X \approx 2^k \|P_{-k}\|_E \quad (k = 0, \pm 1 \pm 2, \dots),$$

*where $P_k(a_j) = (a_{j+k})_{j=-\infty}^{\infty}$ is the shift operator on $E$.*

Proof. We will show that

$$\|\sigma_{2^k} x\|_X = 2^k \|P_{-k}(f(2^j))\|_E \tag{8}$$

for $x \in X$, where $f(t) = \mathcal{K}(t, x; L_1, L_\infty)$. Indeed, we have

$$\|\sigma_{2^k} x\|_X = \|(\mathcal{K}(2^j, x(2^{-k}s); L_1, L_\infty))_j\|_E = \left\| \left( \int_0^{2^j} x^*(2^{-k}s)\, ds \right)_j \right\|_E = 2^k \|P_{-k}(f(2^j))\|_E.$$

In particular, equality (8) yields

$$\|\sigma_{2^k}\| \le 2^k \|P_{-k}\|.$$

Furthermore, since $E$ is interpolatory between $l_\infty$ and $l_\infty(2^{-k})$, there exists a $C_1 > 0$ such that, for each $a = (a_j) \in E$, the sequence $\overline{a} = (\overline{a}_k)$,

$$\overline{a}_k = \mathcal{K}(2^k, a; l_\infty, l_\infty(2^{-j})) = \sup_{j=0,\pm 1,\dots} [\min(1, 2^{k-j})|a_j|],$$

belongs to $E$ too and $\|\overline{a}\|_E \le C_1 \|a\|_E$ [7, § 7].

Observe also that $|a_k| \le \overline{a}_k$ and that

$$\lim_{k \to -\infty} \overline{a}_k = 0 \tag{9}$$

by the monotonicity of $\{\overline{a}_k\}$ and the condition $l_\infty \not\subset E$. For $a \in E$, we introduce the function $f_a(t) = \sum_{k=-\infty}^{\infty} \overline{a}_k \chi_{[2^k,2^{k+1})}(t)$. Since

$$f_a(t_1) \le f_a(t_2), \quad \frac{f_a(t_2)}{t_2} \le 2\frac{f_a(t_1)}{t_1}$$

for $t_1 < t_2$, the function $f_a$ is equivalent to its least concave majorant $\overline{f}_a$ [1, p. 63]. In view of (9), the function $\overline{f}_a$ is absolutely continuous on $[0,\infty)$; therefore,

$$\mathcal{K}(2^k, \overline{f}_a{}'; L_1, L_\infty) = \overline{f}_a(2^k) \approx f_a(2^k) = \overline{a}_k.$$

Consequently, $\overline{f}_a{}' \in X$, and (8) and the monotonicity of $P_k$ imply

$$\begin{aligned} 2^k \|P_{-k}a\|_E \le 2^k \|P_{-k}\overline{a}\|_E \le 2^k \|P_{-k}(\overline{f}_a(2^j))\|_E \\ = \|\sigma_{2^k}(\overline{f}_a{}')\|_X \le \|\sigma_{2^k}\| \, \|(\overline{f}_a(2^j))\|_E \le C_1 C_2 \|\sigma_{2^k}\| \, \|a\|_E, \end{aligned}$$

where $C_2$ is the constant in the preceding equivalence. The proposition is proven.

For simplicity, in what follows we confine ourselves to the case $\psi(t) = t^{1/2}$.

**Theorem 3.** *There exists an r.i. space $Y$ on $[0,1]$, $Y \in J(t^{1/2})$, such that $Y \ne \Lambda(t^{1/2})$ and*

$$\left\| \sum_{k=1}^{\infty} c_k \overline{\chi}_{e_k} \right\|_Y \approx \|(c_k)\|_{l_1} \tag{10}$$

*for all $c_k \ge 0$ and for some system $\{e_k\}_{k=1}^\infty$ of pairwise disjoint subsets of $[0,1]$ satisfying conditions (3) and (4).*

PROOF. There exists a shift-invariant ideal structure on $G$ (i.e., $\sup_{k=0,\pm1,\dots} \|P_k\|_{G\to G} < \infty$) such that, for arbitrary two infinite sets $U \subset \mathbb{Z}$ and $V \subset \mathbb{Z}$, the restrictions $G\big|_U$ and $G\big|_V$ coincide with $l_1\big|_U$ and $c_0\big|_V$ [8, Proposition 1]. Without loss of generality, we can assume that $U \subset \mathbb{Z}_-$, $V \subset \mathbb{Z}_-$, and $\mathbb{Z}_- = \{0,-1,-2,\dots\}$. Denote $E = G(2^{-k/2})$ and $X = (L_1, L_\infty)_E^{\mathcal{K}}$. Then $X$ is an r.i. space on $(0,\infty)$. Since $l_1 \subset G \subset l_\infty$, we have $\Lambda(t^{1/2}) \subset X \subset \mathrm{M}(t^{1/2})$ by (6) and (7); thereby, $\varphi_X(t) \approx t^{1/2}$. It is straightforward that $\|P_k\|_E \approx 2^{k/2}$ (see also [9]). Therefore, by the Proposition, $\|\sigma_{2^k}\|_{X\to X} \approx 2^{k/2}$; whence $\|\sigma_t\|_{X\to X} \approx t^{1/2}$ and $\gamma_X = \alpha_X = \delta_X = \beta_X = 1/2$ by quasiconcavity [1, p. 70].

Let $U = \{-n_k\}_{k=1}^\infty$, where $1 \le n_1 < n_2 < \dots$. Then the system $\{e_k\}_{k=1}^\infty$, $e_k = (2^{-n_{k+1}}, 2^{-n_k}]$, obviously satisfies conditions (3) and (4). If $c_k \ge 0$, then, in virtue of the embedding $\Lambda(t^{1/2}) \subset X$ and relations (5), we have

$$\|y\|_X \le \|y\|_{\Lambda(t^{1/2})} \le A_1 \|(c_k)\|_{l_1}$$

for $y = \sum_{k=1}^\infty c_k \overline{\chi}_{e_k}$. Conversely, if $-n_j < k \le -n_{j-1}$, $j = 2,3,\dots$, then

$$\begin{aligned} \mathcal{K}(2^k, y; L_1, L_\infty) = \int_0^{2^k} y^*(s)\,ds \ge \int_0^{2^k} y(s)\,ds \ge \sum_{i=j}^{\infty} c_i 2^{n_i/2}(2^{-n_i} - 2^{-n_{i+1}}) \\ + c_{j-1} 2^{n_{j-1}/2}(2^k - 2^{-n_j}) \ge 1/2 \cdot 2^{k+n_{j-1}/2} c_{j-1}. \end{aligned}$$

Denote $b = (b_k)$, where $b_k = 2^{-k/2}\mathcal{K}(2^k, y; L_1, L_\infty)$ for $-n_j < k \le -n_{j-1}$, $j = 2,3,\dots$, and $b_k = 0$ for the rest of $k \in \mathbb{Z}$. Then, by the preceding inequality,

$$\|y\|_X \ge \|b\|_E = \|(b_k 2^{-k/2})\|_G \ge \|(b_k 2^{-k/2})\|_{G|_U} = \|(b_k 2^{-k/2})\|_{l_1|_U} \ge \frac{1}{2}\sum_{j=1}^{\infty} |c_j|.$$

Thus,

$$\left\| \sum_{k=1}^{\infty} c_k \overline{\chi}_{e_k} \right\|_X \approx \|(c_k)\|_{l_1}.$$

Denote by $Y$ the space of all measurable functions $x = x(t)$ on $[0,1]$ whose extensions by zero onto $[0,\infty)$ belong to $X$, the space $Y$ being endowed with the norm equal to the norm of the extension in $X$. Then $Y$ is an r.i. space on $[0,1]$, $\gamma_Y = \alpha_Y = \beta_Y = \delta_Y = 1/2$, and (10) holds. The spaces $l_1(2^{-k/2})$ and $E = G(2^{-k/2})$ are interpolation spaces between $l_\infty$ and $l_\infty(2^{-k/2})$ [9]. Therefore, if we suppose that $Y = \Lambda(t^{1/2})$ on $[0,1]$, then, arguing as in the second part of the proof of the Proposition, we can show that $l_1(2^{-k/2})\big|_{\mathbb{Z}_-} = E\big|_{\mathbb{Z}_-}$, i.e., $G\big|_{\mathbb{Z}_-} = l_1|_{\mathbb{Z}_-}$ (see also [10, Proposition 2]). The last fact contradicts the choice of the space $G$. The theorem is proven.

**4.** Denote by $F(\psi)$ the class of those r.i. spaces on $[0,1]$ which have fundamental function $\psi(t)$. We will show that the claim of Theorem 2 fails for $F(\psi)$ even if we replace the equivalence of norms on the cone of functions of the form (1) by their isometric coincidence.

We recall that, for the dilation exponents of the fundamental function and the Boyd indices of an r.i. space, we always have $0 \le \alpha_X \le \gamma_X \le \delta_X \le \beta_X \le 1$. For the most important classes of r.i. spaces (Lorentz spaces, Marcinkiewicz spaces, and Orlicz spaces), we moreover have $\alpha_X = \gamma_X$ and $\beta_X = \delta_X$. Nevertheless, in [11] was presented a first example of an r.i. space on $[0,1]$ for which the last fails. (Since then such r.i. spaces have been called the spaces of nonfundamental type.)

Here we construct an r.i. space $X$ of nonfundamental type on $[0,1]$ for which $\varphi_X(t) = t^{1/2}$ (hence, $\gamma_X = \delta_X = 1/2$), $\alpha_X = 0$, and $\beta_X = 1$, and, at the same time, the norm in $X$ agrees with the norm in $L_2$ on some cone of step functions of the form (1). Henceforth, we denote

$$(x,y) = \int_0^1 x(t)y(t)\,dt, \quad \|x\|_2 = \left(\int_0^1 |x(t)|^2\,dt\right)^{1/2}.$$

For $n \in \mathbb{N}$, $n \ge 2$, $1 \le i \le n$, and $0 < b \le 1$, we put

$$a_0 = 0, \quad a_i = 2^{2n(i-n)} n^{-1}, \quad h_b = b^{-1/2}\chi_{(0,b)}, h_i = h_{a_i}, \quad u_i = h_i \chi_{(a_{i-1},a_i)},$$

$$w_n = n^{-1/2} \sup_{i=1,\dots,n} h_i = n^{-1/2} \sum_{i=1}^{n} u_i, \quad g_n(s) = \sigma_n w_n(s) = w_n(s/n), \quad r_n(s) = \sigma_{n^{-1}} w_n(s) = w_n(sn).$$

First of all, we estimate $\sup_{0<b\le 1}(g_n, h_b)$ and $\sup_{0<b\le 1}(r_n, h_b)$ $(n = 2,3,\dots)$. For $1 < j \le n$, making simple calculations, we obtain

$$\int_0^1 \sum_{i<j} u_i\,dt = \sum_{i=1}^{j-1} a_i^{-1/2}(a_i - a_{i-1}) = \frac{2^{n(j-n)}}{(2^n - 1)n^{1/2}}.$$

Therefore,

$$(w_n, h_j) = n^{-1/2}\left(\sum_{i=1}^{j} \int_0^1 u_i h_j\,dt + \sum_{i=j+1}^{n} \int_0^1 u_i h_j\,dt\right) = n^{-1/2}\left(\int_0^1 u_j h_j\,dt\right.$$

$$\left. + \sum_{i<j} \int_0^1 u_i h_j\,dt\right) \le n^{-1/2}\left(1 + a_j^{-1/2} \int_0^1 \sum_{i<j} u_i\,dt\right) \le n^{-1/2}[1 + (2^n - 1)^{-1}].$$

Now, if $a_i \leq b < a_{i+1}$ $(1 < i < n)$, then

$$(w_n, h_b) \leq \int_0^{a_i} w_n h_i \, dt + n^{-1/2} \int_{a_i}^{b} u_{i+1} h_b \, dt \leq n^{-1/2}[2 + (2^n - 1)^{-1}].$$

The last inequality is obviously valid for $0 < b \leq a_1$ and $a_n \leq b \leq 1$ as well. Thus,

$$(w_n, h_b) \leq n^{-1/2}[2 + (2^n - 1)^{-1}] \tag{11}$$

for all $0 < b \leq 1$. If $c = b/n$ and $d = bn$, then

$$(g_n, h_b) = n \int_0^{1/n} w_n(t) h_b(tn) \, dt = n^{1/2} \int_0^1 w_n(t) h_c(t) \, dt,$$

$$(r_n, h_b) = \frac{1}{n} \int_0^{1/n} w_n(t) h_b(t/n) \, dt = n^{-1/2} \int_0^1 w_n(t) h_d(t) \, dt.$$

Consequently, in view of (11),

$$\sup_{0<b\leq 1} (g_n, h_b) \leq 2 + (2^n - 1)^{-1} \leq 3, \tag{12}$$

$$\sup_{0<b\leq 1} (r_n, h_b) \leq \frac{1}{n}[2 + (2^n - 1)^{-1}] \leq \frac{3}{n}. \tag{13}$$

We will define a cone $V$ and thus a space $X$. Take a sequence of natural numbers by putting $n_1 = 2$ and

$$n_{m+1} = 2^{2n_m(n_m - 1)} n_m \quad (m = 1, 2, \ldots). \tag{14}$$

If $(a_i^m)_{i=0}^{n_m}$ is the collection of points constructed from the number $n_m$ as above, then, by (14), their union over $m = 1, 2, \ldots$, enumerated in a natural way, forms a sequence that vanishes monotonously and the system of all the intervals $(a_{i-1}^m, a_i^m]$ form a partition of $[0, 1/2]$. Denote by $V$ the cone of functions of the form

$$v(t) = c_0 \chi_{[1/2,1]}(t) + \sum_{m=1}^{\infty} \sum_{i=2}^{n_m} c_i^m \chi_{i,m}(t),$$

where

$$\chi_{i,m}(t) = \chi_{(a_{i-1}^m, a_i^m]}(t) \quad (m = 1, 2, \ldots; \ i = 2, \ldots n_m),$$

$$0 \leq c_0 \leq c_{n_1}^1 \leq c_{n_1-1}^1 \leq \ldots \leq c_2^1 \leq c_{n_2}^2 \leq \ldots \leq c_2^2 \leq \ldots \leq c_{n_k}^k \leq \ldots \leq c_2^k \leq \ldots .$$

If $Q = V \cup \{h_b, 0 < b \leq 1\}$, then, by definition, the space $X$ consists of all measurable functions $x = x(t)$ on $[0, 1]$ for which

$$\|x\|_X = \sup_{\substack{q \in Q, \\ \|q\|_2 \leq 1}} \int_0^1 x^*(t) q(t) \, dt < \infty.$$

The properties of a rearrangement of a function [1, pp. 88–106] imply that $X$ is an r.i. space on $[0, 1]$. Moreover, $\|q\|_X = \|q\|_2$ for $q \in Q$. Therefore,

$$\varphi_X(t) = \|\chi_{(0,t)}\|_X = t^{1/2} \|h_t\|_2 = t^{1/2},$$

and, for $w_m = w_{n_m}$, we have $\|w_m\|_X = \|w_m\|_2 = 1 - 2^{-2n_m}$. Hence, $L_2 \subset X \subset \mathrm{M}(t^{1/2})$ and

$$3/4 \le \|w_m\|_X = \|w_m\|_2 \le 1 \tag{15}$$

for all $m = 1, 2, \ldots$.

Prove that $\alpha_X = 0$ and $\beta_X = 1$. To this end, it suffices in view of (15) to check that $\|g_m\|_X \approx 1$ and $\|r_m\|_X \approx n_m^{-1}$ ($g_m = g_{n_m}$, $r_m = r_{n_m}$). In virtue of relations (12) and (13) and the definition of $X$, the last fact is equivalent to uniform boundedness of $(g_m, v)$ and $n_m(r_m, v)$ for all $v \in V$, $\|v\|_2 \le 1$, and $m \in \mathbb{N}$.

Thus, let $v \in V$ and $m \in \mathbb{N}$. Represent $v$ as $v = v_1 + v_2 + v_3$, with

$$v_1 = \sum_{j=1}^{m-1} \sum_{i=2}^{n_j} c_i^j \chi_{i,j} + c_0 \chi_{[1/2,1]}, \quad v_2 = \sum_{i=2}^{n_m} c_i^m \chi_{i,m}, \quad v_3 = v - v_1 - v_2.$$

Since $n_m a_{n_m-1}^m \le a_1^{m-1}$ by (14); therefore, for $n = n_m$, we have

$$(g_m, v_1) = \int_{a_1^{m-1}}^{1} g_m v_1 \, dt \le n^{-1/2} \int_{a_1^{m-1}}^{1} u_n^m(t/n) v_1(t) \, dt \le n^{-1/2} \|\sigma_n(u_n^m)\|_2 \|v_1\|_2 \le \|u_n^m\|_2 \|v_1\|_2.$$

However, $\|u_n^m\|_2 \le 1$; consequently,

$$(g_m, v_1) \le 1. \tag{16}$$

If $\tau = 1/n_{m+1}$, then

$$(g_m, v_3) = \int_0^\tau g_m v_3 \, dt = \int_0^\tau w_m v_3 \, dt,$$

since the function $w_m$ is constant on $(0, \tau)$. Consequently,

$$(g_m, v_3) \le \|w_m\|_2 \|v_3\|_2 \le 1. \tag{17}$$

Finally, we estimate $(g_m, v_2)$. Represent $g_m$ as $g_m = w_m' + w_m''$, where

$$w_m' = n^{-1/2} \sum_{i=1}^{n} a_i^{-1/2} \chi_{(na_{i-1}, b_{i-1})}, \quad w_m'' = n^{-1/2} \sum_{i=1}^{n} a_i^{-1/2} \chi_{(b_{i-1}, na_i)},$$
$$b_{i-1} = na_{i-1} + a_i - a_{i-1}, \quad n = n_m, \quad a_i = a_i^m.$$

Since the functions $w_m'$ and $w_m$ are equimeasurable, from (15) we infer $\|w_m'\|_2 = \|w_m\|_2 \le 1$. Moreover, it is easy to check that $a_i^{-1/2}(a_i - a_{i-1}) \le 2^{-n}(a_{i+1} - a_i)^{1/2}$ for $1 \le i \le n-1$. Therefore,

$$\begin{aligned}
(g_m, v_2) &= (w_m', v_2) + (w_m'', v_2) \le \|w_m'\|_2 \|v_2\|_2 \\
&+ n^{-1/2} \sum_{i=1}^{n-1} c_{i+1}^m a_i^{-1/2} (n-1)(a_i - a_{i-1}) \le 1 + \frac{n-1}{2^n \sqrt{n}} \sum_{i=1}^{n-1} c_{i+1}^m (a_{i+1} - a_i)^{1/2} \\
&\le 1 + \frac{n-1}{2^n \sqrt{n}} \left[ \sum_{i=1}^{n-1} \left(c_{i+1}^m\right)^2 (a_{i+1} - a_i) \right]^{1/2} \left( \sum_{i=1}^{n-1} 1 \right)^{1/2} \le 1 + \frac{n-1}{2^n} \|v_2\|_2 \le 2;
\end{aligned}$$

i.e.,

$$(g_m, v_2) \le 2. \tag{18}$$

As a result, from (16)–(18) we obtain

$$(g_m, v) \le 4 \tag{19}$$

for all $m \in \mathbb{N}$ and $v \in V$, $\|v\|_2 \le 1$. Arguing in the same way, we can show that $(w_m, \sigma_{n_m} v) \le 3$ for $m \in \mathbb{N}$ and $v \in V$, $\|v\|_2 \le 1$. Therefore,

$$(r_m, v) = \frac{1}{n_m}(w_m, \sigma_{n_m} v) \le \frac{3}{n_m}. \tag{20}$$

As has already been said, relations (19) and (20) imply $\alpha_X = 0$ and $\beta_X = 1$.

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