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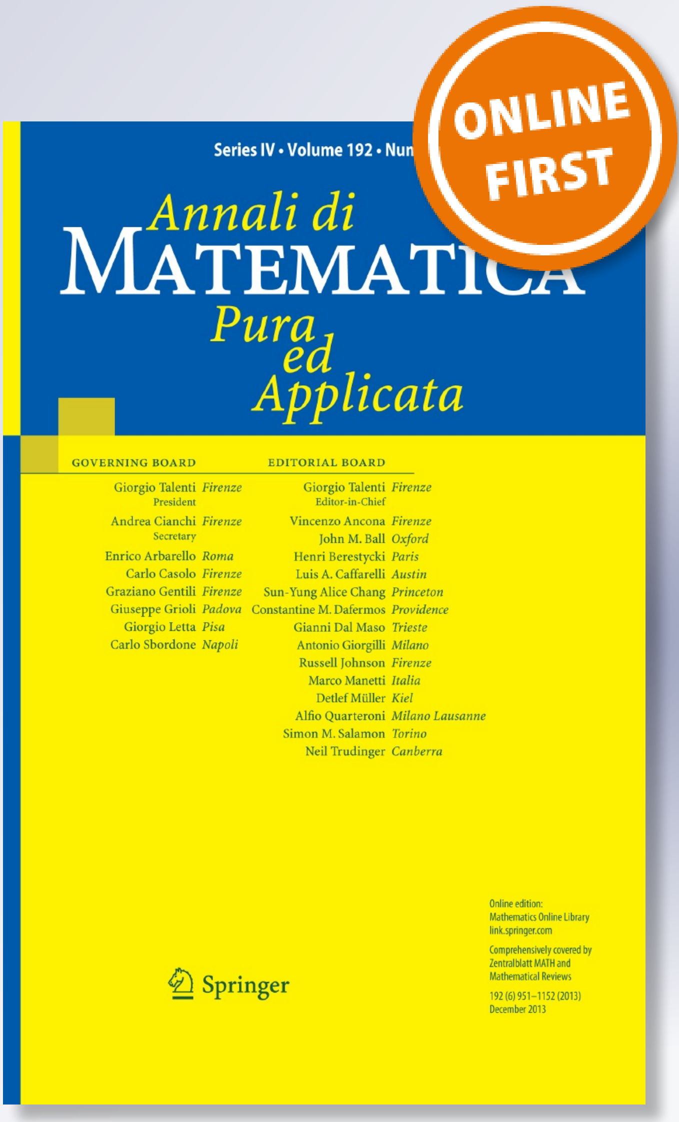

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Abstract We prove that the local version of Khintchine inequality holds in an rearrangement invariant function space X on [0,1] if and only if the lower dilation index of the fundamental function of X is positive. A further characterization is given, based on the uniform behavior in X of the dilations of the logarithmic function. For this, a study of the space of functions acting as multiplication operators in X for the tails of Rademacher series is carried out.

Keywords Rademacher functions · Rearrangement invariant space · Khintchine inequality

Mathematics Subject Classification (2000) Primary 46E35 · 46E30; Secondary 47G10

1 Introduction

The classical Khintchine inequality states, for $0 < p < \infty$, that there exist constants $A_p, B_p > 0$ such that

$$A_p\left(\sum_{i=1}^{\infty} a_i^2\right)^{1/2} \le \left(\int_0^1 \Big|\sum_{i=1}^{\infty} a_i r_i(t)\Big|^p \,\mathrm{d}t\right)^{1/p} \le B_p\left(\sum_{i=1}^{\infty} a_i^2\right)^{1/2},$$

for every $(a_i) \in \ell^2$, where $(r_i)_1^\infty$ are the Rademacher functions. Rodin and Semenov gave the full extent of the inequality by characterizing those rearrangement invariant (r.i.) spaces X for

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which Khintchine inequality holds [15]. The inequality has had a wide range of applications all throughout analysis. Having applications in mind, it is of interest considering a local version of Khintchine inequality, that is, the case when the underlying set is $E \subset [0,1]$ other than $[0,1]$. For example, in the case when E is a dyadic interval, e.g., $E = \Delta_n^k$, for $\Delta_n^k = [(k-1)2^{-n}, k2^{-n}]$, with $n \in \mathbb{N}$ and $1 \le k \le 2^n$, the situation is still somewhat similar to that of the full interval $[0,1]$, due to the fact that the tail Rademacher sequence $(r_i)_{i=n+1}^{\infty}$ behaves on Δ_n^k as does the full sequence on $[0,1]$. The question is what happens in the case when E is an arbitrary measurable set with positive measure.

A local version of Khintchine inequality was first given by Zygmund for $L^2([0,1])$, showing that there exist constants $A_2', B_2' > 0$ so that for any measurable set $E \subset [0,1]$ with $m(E) > 0$ there exists $N := N(E)$ such that

$$A_2'\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \le \left(\frac{1}{m(E)}\int_E \Big|\sum_{i=N}^{\infty} a_i r_i(t)\Big|^2 \,\mathrm{d}t\right)^{1/2} \le B_2'\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2},$$

for every $(a_i) \in \ell^2$, [20], see also [21, Lemma 5.8.3]. The result was extended by Sagher and Zhou to $L^p([0,1])$, for all finite values of $p > 0$, [17], and recently to spaces of exponential integrability, [8,19].

We say that a r.i. space X satisfies the *local version of Khintchine inequality* if there exist constants $\alpha, \beta > 0$ such that for any measurable set $E \subset [0,1]$ with $m(E) > 0$ there exists $N := N(E)$ such that

$$\alpha\varphi_X(m(E))\Big\|\sum_{i=N}^{\infty} a_i r_i\Big\|_X \le \Big\|\sum_{i=N}^{\infty} a_i r_i \chi_E\Big\|_X \le \beta\varphi_X(m(E))\Big\|\sum_{i=N}^{\infty} a_i r_i\Big\|_X,$$

for every $(a_i) \in \ell^2$ such that $\sum_{i=N}^{\infty} a_i r_i \in X$, where $\varphi_X(t) := \|\chi_{[0,t]}\|_X$ is the fundamental function of X.

The main result of the paper is that the local version of Khintchine inequality holds in X if and only if either $X = L_\infty$ or

$$\gamma_{\varphi_X} > 0,$$

where the γ_{φ_X} is the lower dilation index of the fundamental function φ_X, Theorem 5.2 and Remark 5.4. This condition is fulfilled by a large number of spaces, including all spaces which are not "close" to L^∞, see Example 5.5. In particular, all separable Orlicz spaces and all spaces having positive lower Boyd index α_X satisfy it. Furthermore, we will see that, for $X \ne L_\infty$, the local version of Khintchine inequality holds in X if and only if the following inequality holds

$$\alpha\varphi_X(m(E))\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \le \Big\|\sum_{i=N}^{\infty} a_i r_i \chi_E\Big\|_X \le \beta\varphi_X(m(E))\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2},$$

for every $(a_i) \in \ell^2$. Note that for $X = L^p([0,1])$, we have $\varphi_X(t) = t^{1/p}$, so, in particular, we recover Zygmund's (and Sagher and Zhou's) result.

A further characterization of the local Khintchine inequality is possible, not in terms of an index but in terms of the uniform behavior of the dilations of the logarithmic function. Namely, the local version of Khintchine inequality holds in X if and only if $\log^{1/2}(2/t)$ belongs to the separable part of X and there exists a constant $\beta > 0$ such that, for all $u \in (0,1]$,

$$\Big\|\log^{1/2}\left(\frac{2u}{t}\right)\chi_{[0,u]}\Big\|_X \le \beta\|\chi_{[0,u]}\|_X,$$

Theorem 5.2. In order to establish this characterization, we need to study in detail the behavior of partial sums and tails of Rademacher series in X. For this, we consider the space of all measurable functions $f : [0, 1] \to \mathbb{R}$, which define by multiplication a bounded linear operator on X when acting over the tails of Rademacher series. This is the tail Rademacher multiplicator space, denoted by $\Lambda_T(X)$ which, for step dyadic functions of order n, amounts to considering expressions of the form

$$\sup\left\{\Big\|f\cdot\sum_{i=n+1}^{\infty}a_ir_i\Big\|_X : \Big\|\sum_{i=n+1}^{\infty}a_ir_i\Big\|_X\le 1\right\}.$$

The space $\Lambda_T(X)$ is related to the problem of r.i. spaces containing the Rademacher multiplicator space $\Lambda(X)$ (see below for the definition). In fact, $\Lambda_T(X)$ can be seen as the version of $\Lambda(X)$ which is obtained by suitable replacement of Rademacher series with their tails.

The paper is organized as follows. In Sect. 2, we present firstly general preliminaries on functions spaces, and secondly facts on Rademacher functions, their behavior in r.i. function spaces, and the spaces $\Lambda(X)$ of Rademacher multiplicators which they generate. Section 3 is devoted to establishing the existence of the Rademacher multiplicator space $\Lambda_T(X)$. In particular, it is shown that, when $\log^{1/2}(2/t) \in X_0$, the space $\Lambda_T(X)$ is an r.i. space, which is larger than $\Lambda(X)$ and coincides with it when the latter is r.i., Theorem 3.6. The fundamental function of $\Lambda_T(X)$, which will be used in the second characterization of the local Khintchine inequality, is given in Remark 3.7. In Sect. 4, the space $\Lambda_T(X)$ is compared with other related function spaces, Theorem 4.1. In particular, the equality $\Lambda_T(X) = X$ holds if and only if the lower Boyd index α_X of X is positive, Theorem 4.3. Section 5 is devoted to proving the main result of the paper, Theorem 5.2, characterizing the validity in X of the above-mentioned local version of Khintchine inequality.

2 Preliminaries

2.1 Function spaces

Throughout the paper an rearrangement invariant (r.i.) space X is a Banach space of classes of measurable functions on [0,1] such that if $g^* \le f^*$ and $f \in X$ then $g \in X$ and $\|g\|_X \le \|f\|_X$. Here f^* is the decreasing rearrangement of f, that is, the right continuous inverse of its distribution function: $m(\tau) := m\{t \in [0, 1] : |f(t)| > \tau\}$, where m is the Lebesgue measure on [0,1]. Functions f and g are said to be equimeasurable if $m_f(\tau) = m_g(\tau)$, for all $\tau > 0$. The associated (or Köthe dual) space of X is the space X' of all functions g such that $\int_0^1 |f(t)g(t)|\,dt < \infty$, for every $f \in X$. It is an r.i. space. The space X' is a subspace of the topological dual X^* and $X' = X^*$ if and only if X is separable. The space X is said to have the Fatou property when $X = X''$, where $X'' = (X')'$. This is equivalent to the following: from $f_n \in X$, $n \in \mathbb{N}$, $\sup_n \|f_n\|_X < \infty$, and $f_n \to f$ a.e., it follows that $f \in X$ and $\|f\|_X \le \liminf_n \|f_n\|_X$. A weaker concept is that of order subcontinuous norm, i.e., X has order subcontinuous norm if from $f_n \in X$, $n \in \mathbb{N}$, $\sup_n \|f_n\|_X < \infty$, $f \in X$, and $f_n \to f$ a.e., it follows that $\|f\|_X \le \liminf_n \|f_n\|_X$. It is known that the norm of X is order subcontinuous if and only if X is isometrically embedded into X'' [11, Theorem 6.1.6].

We denote by X_0 the closure of L^∞ in X. If $X \ne L^\infty$, then X_0 coincides with the absolutely continuous part of X, that is, the set of all functions $f \in X$ such that $\lim_{m(A)\to 0} \|f\chi_A\|_X = 0$, where χ_A denotes the characteristic function of the set $A \subset [0, 1]$. The fundamental function of X is the function $\varphi_X(t) := \|\chi_{[0,t]}\|_X$.

For each $t > 0$, the dilation operator $\sigma_t f(s) := f(s/t)\chi_{[0,1]}(s/t)$, $s \in [0,1]$, is bounded on any r.i. space X. The lower Boyd index of X is

$$\alpha_X := \lim_{t\to 0^+} \frac{\log \|\sigma_t\|_{X\to X}}{\log t}.$$

Important examples of r.i. spaces are Lorentz, Orlicz, and Marcinkiewicz spaces. For a quasi-concave function $\varphi\colon [0,1] \to [0,+\infty)$ (i.e., φ increases, $\varphi(t)/t$ decreases and $\varphi(0) = 0$), the Marcinkiewicz space $M(\varphi)$ consists of all measurable functions f on [0,1] for which the norm

$$\|f\|_{M(\varphi)} := \sup_{0<t\le 1} \frac{\varphi(t)}{t} \int_0^t f^*(s)\,ds < \infty.$$

Let M be an Orlicz function, that is, an increasing convex function on $[0,\infty)$ with $M(0) = 0$. The Orlicz space L_M consists of all measurable functions f on [0,1] for which the norm

$$\|f\|_{L_M} = \inf\left\{\lambda > 0 : \int_0^1 M\left(\frac{|f(t)|}{\lambda}\right) \mathrm{d}t \le 1\right\} < \infty.$$

Recall that if ψ is a positive function defined on [0,1], then its lower dilation index (see [12, p. 54]) is

$$\gamma_\psi := \lim_{t\to 0^+} \frac{\log \mathcal{M}_\psi(t)}{\log t}, \quad \text{for} \quad \mathcal{M}_\psi(t) := \sup_{0<s\le 1} \frac{\psi(st)}{\psi(s)}.$$

Throughout the paper, $A \asymp B$ means that there exist constants $C > 0$ and $c > 0$ such that $c\cdot A \le B \le C\cdot A$. For convenience of computations, all logarithms will be considered with base 2.

For any undefined notion regarding r.i. spaces, we refer the reader to the monographs [6,12,13].

2.2 Rademacher functions and Rademacher multiplicator spaces

The Rademacher functions are $r_n(t) := \operatorname{sign} \sin(2^n \pi t)$, $t \in [0,1]$, $n \in \mathbb{N}$. Let $\mathcal{R}$ denote the set of all functions of the form $\sum_{n=1}^{\infty} a_n r_n$, where the series converges a.e., that is, with $(a_n) \in \ell^2$ [21, Theorem V.8.2]. For an r.i. space X on [0,1], let $\mathcal{R}(X)$ be the closed linear subspace of X given by $\mathcal{R} \cap X$. For $X = L^p$, $1 \le p < \infty$, the Khintchine inequality shows that $\mathcal{R}(X)$ is isomorphic to ℓ^2. If $X = L^\infty$, then $\mathcal{R}(X) \approx \ell^1$ [15]. The Orlicz space L_N, associated with the function $N(t) := \exp(t^2) - 1$, is of major importance in this study. The previously mentioned result of Rodin and Semenov shows that $\mathcal{R}(X) \approx \ell^2$ if and only if $(L_N)_0 \subset X$, [15]. Hence, for r.i. spaces X satisfying this condition, we have

$$\left\| \sum_{n=1}^{\infty} a_n r_n \right\|_X \asymp \|(a_n)\|_2.$$

The fundamental function of L_N is (equivalent to) $\varphi(t) = \log^{-1/2}(2/t)$. Since $N(t)$ increases very rapidly (see [14] and [16]), L_N is also the set of all functions satisfying

$$\|f\|_{L_N} \asymp \sup_{0<t\le 1} f^*(t) \log^{-1/2}(2/t) < \infty,$$

which implies that for every $0 < t \le 1$ we have

$$f^*(t) \le C\, \|f\|_{L_N}\ \log^{1/2}(2/t).$$

Hence, for an r.i. space X, the inclusion $L_N \subset X$ holds if and only if $\log^{1/2}(2/t) \in X$. If $(a_n) \in \ell^2$, then $\sum_{n=1}^{\infty} a_n r_n \in L_N$ and, therefore, we have that, for a universal constant $\gamma \ge 1$,

$$\left(\sum_{n=1}^{\infty} a_n r_n\right)^*(t) \le \gamma\ \|(a_n)\|_2 \cdot \log^{1/2}(2/t), \quad 0 < t \le 1. \tag{2.1}$$

The *Rademacher multiplicator space* of X was firstly considered in [9]. It is the space $\Lambda(X)$ of all measurable functions $f\colon [0,1] \to \mathbb{R}$ such that $f \sum_{n=1}^{\infty} a_n r_n \in X$, for every $\sum a_n r_n \in \mathcal{R}(X)$. It is a Banach function space on [0,1] when endowed with the norm

$$\|f\|_{\Lambda(X)} := \sup\left\{\left\| f \sum_{n=1}^{\infty} a_n r_n \right\|_X : \sum_{n=1}^{\infty} a_n r_n \in X, \left\|\sum_{n=1}^{\infty} a_n r_n\right\|_X \le 1\right\},$$

and it can be seen as the space of operators from $\mathcal{R}(X)$ into the whole space X given by multiplication by a measurable function. The space $\Lambda(X)$ is not r.i. whenever $\gamma_{\varphi_X} > 0$, [2]. This result motivates the study of the symmetric kernel $\mathrm{Sym}\,(X)$ of the space $\Lambda(X)$, that is, the largest r.i. space embedded into $\Lambda(X)$: we have that, if $L_N \subset X$, then $\mathrm{Sym}\,(X)$ is the r.i. space with norm $\|f\| \asymp \|f^*(t)\log^{1/2}(2/t)\|_{X''}$, [2] and [4]. The opposite situation is when $\Lambda(X)$ is r.i., see examples in [9], [10], and [2]. The simplest case, when $\Lambda(X) = L^\infty$, holds if and only if $\log^{1/2}(2/t) \notin X_0$, [1] and [3]. Necessary conditions and sufficient conditions for $\Lambda(X)$ being an r.i. space different from L^∞ were studied in [4].

3 The tail Rademacher multiplicator space $\Lambda_T(X)$

Let X be an r.i. space on [0,1]. We define the *tail Rademacher multiplicator space* of X. Denote by $\mathcal{D}_n$, for $n \in \mathbb{N}$, the set of all step functions of the form

$$f = \sum_{k=1}^{2^n} c_k \chi_{\Delta_k^n}, \quad c_k \in \mathbb{R}, \tag{3.1}$$

where $\Delta_k^n = [(k-1)2^{-n}, k2^{-n}]$ are the dyadic intervals of order n, for $n \in \mathbb{N}$, and $1 \le k \le 2^n$. Set $\mathcal{D} := \cup_n \mathcal{D}_n$. Then, $\mathcal{D}$ is a linear subspace of X.

We define on $\mathcal{D}$ the following functional: if $f \in \mathcal{D}_n$ then

$$\|f\|^{\circ}_{\Lambda_T(X)} := \sup\left\{\left\| f \cdot \sum_{i=n+1}^{\infty} a_i r_i \right\|_X : \left\|\sum_{i=n+1}^{\infty} a_i r_i\right\|_X \le 1\right\}. \tag{3.2}$$

The definition is independent of the representation of f in the form (3.1). Indeed, if $f \in \mathcal{D}_n$, then $f \in \mathcal{D}_{n+1}$. On each set Δ_k^n, where f is constant, the functions $\sum_{i=n+1}^{\infty} a_i r_i$ and $\sum_{i=n+2}^{\infty} a_{i-1} r_i$ have the same distribution. Since X is r.i., this implies that $\|f \cdot \sum_{i=n+1}^{\infty} a_i r_i\|_X = \|f \cdot \sum_{i=n+2}^{\infty} a_{i-1} r_i\|_X$. This fact also establishes that $\|\cdot\|^{\circ}_{\Lambda_T(X)}$ is a norm on $\mathcal{D}$.

Moreover, from the definition, it follows that

$$\|f\|^{\circ}_{\Lambda_T(X)} \le \|f\|_{\Lambda(X)}, \quad f \in \mathcal{D}. \tag{3.3}$$

We denote by $\Lambda_T(\mathcal{R}, X)^\circ$ ($\Lambda_T(X)^\circ$ in short) the completion of $\mathcal{D}$ with respect to the norm (3.2).

Firstly, suppose that $\log^{1/2}(2/t) \notin X_0$. Then, by Astashkin and Curbera [3] (see formula (2) in the proof of Theorem 1), there exists a constant $c_0 > 0$, depending only on X, such that for every $m \geq 0$, there exists $n_0 \geq 1$ such that, if $n \geq n_0$ and Δ is an arbitrary dyadic interval of order m, we have

$$\frac{\left\| \chi_\Delta \sum_{m+1}^{m+n} r_i \right\|_X}{\left\| \sum_{m+1}^{m+n} r_i \right\|_X} \geq c_0. \tag{3.4}$$

This implies that $\|\chi_\Delta\|^\circ_{\Lambda_T(X)} \geq c_0$ for every dyadic interval Δ. Then, it is easy to check that each Cauchy sequence $\{f_k\}$ of functions of the form (3.1) in the $\Lambda_T(X)^\circ$-norm is a Cauchy sequence with respect to the uniform convergence on $[0, 1]$. In fact, if we suppose that $\lim_{n\to\infty} \sup_{m>n} \|f_n - f_m\|_{L_\infty} \geq \delta$ for some $\delta > 0$, then there exist $n_k \in \mathbb{N}$, $n_k \uparrow \infty$, $m_k > n_k$, and dyadic intervals Δ_k such that $|f_{n_k} - f_{m_k}| \geq \delta\chi_{\Delta_k}$ for all $k \in \mathbb{N}$. Hence, $\|f_{n_k} - f_{m_k}\|^\circ_{\Lambda_T(X)} \geq c_0\delta$ $(k = 1, 2, \ldots)$, which contradicts the Cauchy condition with respect to $\Lambda_T(X)^\circ$-norm. Therefore, the space $\Lambda_T(X)^\circ$ coincides with the set of all uniform limits of sequences from $\mathcal{D}$. It is not hard to check that $\Lambda_T(X)^\circ$ consists of all functions f satisfying (after, possibly, a change on a set of measure zero) the following property: for an arbitrary $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that the oscillation of f on every dyadic interval of order n does not exceed ε, that is,

$$\sup_{t_1, t_2 \in \Delta_k^n} |f(t_1) - f(t_2)| \leq \varepsilon, \quad k = 1, 2, \ldots, 2^n.$$

It is clear that in this case, $\Lambda_T(X)^\circ$ is not a Banach lattice and that

$$C([0, 1]) \cup \mathcal{D} \subsetneq \Lambda_T(X)^\circ \subsetneq L_\infty([0, 1]).$$

Consider now the more interesting case when $\log^{1/2}(2/t) \in X_0$. Then, $\mathcal{R}(X) \approx \ell^2$ (see Preliminaries) and so, for $f \in \mathcal{D}_n$, we have

$$\|f\|^\circ_{\Lambda_T(X)} \asymp \sup\left\{ \left\| f \cdot \sum_{i=n+1}^\infty a_i r_i \right\|_X : \sum_{i=n+1}^\infty a_i^2 \leq 1 \right\}. \tag{3.5}$$

Moreover, in this case, there is another useful equivalent expression for the $\Lambda_T(X)^\circ$-norm.

Lemma 3.1 *Let X be an r.i. space on* $[0, 1]$ *such that* $\log^{1/2}(2/t) \in X_0$. *Then for all* $n \in \mathbb{N}$ *and* $f \in \mathcal{D}_n$ *of the form* (3.1)

$$\|f\|^\circ_{\Lambda_T(X)} \asymp \left\| \sum_{k=1}^{2^n} c_k \log^{1/2}\left(\frac{2}{2^n t - k + 1} \right) \cdot \chi_{\Delta_k^n} \right\|_X$$

with constants independent of n and f.

Proof Let f be in the form (3.1). Taking into account that $\log^{1/2}(2/t) \in X_0$ and inequality (2.1), for every $k = 1, 2, \ldots, 2^n$ and any sequence $\{a_i\}_{i=n+1}^\infty$ such that $\sum_{i=n+1}^\infty a_i^2 \leq 1$, we have

$$\begin{aligned} \left(\sum_{i=n+1}^\infty a_i r_i \cdot \chi_{\Delta_k^n} \right)^* (t) &= \left(\sum_{i=1}^\infty a_{i+n} r_i \right)^* (2^n t) \chi_{[0,2^{-n}]}(t) \\ &\leq \gamma \log^{1/2}(2^{-n+1} t^{-1}) \chi_{[0,2^{-n}]}(t). \end{aligned}$$

Since X is r.i. and the intervals Δ_k^n, for $k = 1, 2, \dots, 2^n$, are disjoint, the above inequality and (3.5) imply that

$$\|f\|^{\circ}_{\Lambda_T(X)} \le \gamma \left\| \sum_{k=1}^{2^n} c_k \log^{1/2}\left(\frac{2}{2^n t - k + 1}\right) \cdot \chi_{\Delta_k^n} \right\|_X.$$

For the reverse inequality, we follow an argument from the proof of Astashkin and Curbera [4, Proposition 3.3]. Firstly, by (3.5), there exists $b_1 > 0$ such that

$$\begin{aligned} \|f\|^{\circ}_{\Lambda_T(X)} &\ge b_1 \sup_{m\in\mathbb{N}} \left\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k} \cdot \sum_{i=n+1}^{n+m} \frac{1}{\sqrt{m}} r_i \right\|_X \\ &= b_1 \sup_{m\in\mathbb{N}} \left\| \sum_{k=1}^{2^n} c_k \chi_{\Delta_n^k}(t) \cdot \Big(\sum_{i=1}^{m} \frac{1}{\sqrt{m}} r_i\Big)^*(2^n t - k + 1) \right\|_X. \end{aligned}$$

Since, by the central limit theorem (see [15] or [13, Theorem 2.b.4]),

$$\lim_{m\to\infty} \left(\frac{1}{\sqrt{m}} \sum_{i=1}^{m} r_i\right)^*(t) \ge \log^{1/2}\left(\frac{2}{e\sqrt{2\pi}\, t}\right), \quad 0 < t \le \frac{2}{e\sqrt{2\pi}},$$

it follows that

$$\|f\|^{\circ}_{\Lambda_T(X)} \ge b_2 \left\| \sum_{k=1}^{2^n} c_k \log^{1/2}\left(\frac{2}{2^n t - k + 1}\right) \cdot \chi_{\Delta_k^n}(t) \right\|_{X''}.$$

By assumption, $\log^{1/2}(2/t) \in X_0$. Therefore, since $(X'')_0 = X_0$ isometrically, we can change the norm of X'' by the norm of X, i.e., we have

$$\|f\|^{\circ}_{\Lambda_T(X)} \ge b_2 \left\| \sum_{k=1}^{2^n} c_k \log^{1/2}\left(\frac{2}{2^n t - k + 1}\right) \cdot \chi_{\Delta_k^n}(t) \right\|_{X}.$$

□

Let X be an r.i. space on $[0, 1]$ such that $\log^{1/2}(2/t) \in X_0$. For all $n \in \mathbb{N}$ and $f \in \mathcal{D}_n$ of the form (3.1), we set

$$\|f\|_{\Lambda_T(X)} := \left\| \sum_{k=1}^{2^n} c_k \log^{1/2}\left(\frac{2}{2^n t - k + 1}\right) \cdot \chi_{\Delta_k^n} \right\|_X. \tag{3.6}$$

The definition is independent of the representation of f in the form (3.1). Indeed, if $f \in \mathcal{D}_n$, then $f \in \mathcal{D}_{n+1}$ and

$$f = \sum_{k=1}^{2^{n+1}} m_k \chi_{\Delta_k^{n+1}}, \quad m_{2k} = m_{2k-1} = c_k \ (1 \le k \le 2^n).$$

Thus, the distribution of $c_k \log^{1/2}\left(\frac{2}{2^n t - k+1}\right) \cdot \chi_{\Delta_k^n}$ is the same as that of

$$m_{2k-1} \log^{1/2}\left(\frac{2}{2^{n+1} t - (2k-1) + 1}\right) \cdot \chi_{\Delta_{2k-1}^{n+1}} + m_{2k} \log^{1/2}\left(\frac{2}{2^{n+1} t - 2k + 1}\right) \cdot \chi_{\Delta_{2k}^{n+1}}.$$

Since X is r.i., this implies that

$$\Big\| \sum_{k=1}^{2^n} c_k \log^{1/2}\left(\frac{2}{2^n t - k + 1}\right) \cdot \chi_{\Delta_k^n} \Big\|_X = \Big\| \sum_{k=1}^{2^{n+1}} m_k \log^{1/2}\left(\frac{2}{2^{n+1} t - k + 1}\right) \cdot \chi_{\Delta_k^{n+1}} \Big\|_X.$$

This fact establishes also that $\|\cdot\|_{\Lambda_T(X)}$ is a norm on $\mathcal{D}$.

By Lemma 3.1, the $\Lambda_T(X)^\circ$- and $\Lambda_T(X)$-norms are equivalent on $\mathcal{D}$. Therefore, $\Lambda_T(X)^\circ$ is the completion of $\mathcal{D}$ with respect to the $\Lambda_T(X)$-norm as well.

Lemma 3.2 *Let X be an r.i. space on* $[0, 1]$ *such that* $\log^{1/2}(2/t) \in X_0$. *Let $E \subset [0, 1]$ be a measurable set with positive measure. Then $\chi_E \in \Lambda_T(X)^\circ$ and*

$$\|\chi_E\|_{\Lambda_T(X)} = \Big\| \log^{1/2}\left(\frac{2m(E)}{t}\right) \chi_{[0,m(E)]} \Big\|_X. \tag{3.7}$$

Proof Since E is measurable, there exists an increasing sequence of integers $\{n_k\}_{k=1}^\infty$ such that for some unions of dyadic sets of order n_k,

$$E_k = \bigcup_{j \in J_k} \Delta_j^{n_k}, \quad J_k \subset \{1, 2, \dots, 2^{n_k}\},$$

we have $m(E \triangle E_k) \le 1/k$, for $k \in \mathbb{N}$. The sequence $\{\chi_{E_k}\}$ is in $\mathcal{D}$. For $k \ge l$, we have

$$\chi_{E_k} - \chi_{E_l} = \sum_{j \in J_k} \chi_{\Delta_j^{n_k}} - \sum_{j \in J_l} \chi_{\Delta_j^{n_l}} = \sum_{j \in J_{k,l}} \pm \chi_{\Delta_j^{n_k}},$$

where $J_{k,l}$ is a subset of $\{1, 2, \dots, 2^{n_k}\}$. Moreover,

$$\sum_{j \in J_{k,l}} m(\Delta_j^{n_k}) = m(E_k \triangle E_l) \le m(E_k \triangle E) + m(E \triangle E_l) \le \frac{2}{l}.$$

Then, using formula (3.6), we have

$$\begin{aligned}
\|\chi_{E_k} - \chi_{E_l}\|_{\Lambda_T(X)} &= \Big\| \sum_{j \in J_{k,l}} \chi_{\Delta_j^{n_k}} \Big\|_{\Lambda_T(X)} \\
&= \Big\| \sum_{j \in J_{k,l}} \log^{1/2}\left(\frac{2}{2^{n_k} t - j + 1}\right) \cdot \chi_{\Delta_j^{n_k}} \Big\|_X \\
&\le \Big\| \log^{1/2}\left(\frac{4}{lt}\right) \cdot \chi_{[0,2/l]} \Big\|_X.
\end{aligned}$$

Since $\log^{1/2}(2/t) \in X_0$, then

$$\lim_{l \to \infty} \Big\| \log^{1/2}\left(\frac{4}{lt}\right) \cdot \chi_{[0,2/l]} \Big\|_X = 0.$$

This implies that $\{\chi_{E_k}\}$ is a Cauchy sequence for the $\Lambda_T(X)$-norm. Since $\{\chi_{E_k}\}$ tends to χ_E in measure, it follows that $\chi_E \in \Lambda_T(X)$ and $\|\chi_E - \chi_{E_k}\|_{\Lambda_T(X)} \to 0$.

Moreover, using again (3.6), we have

$$\begin{aligned}
\|\chi_{E_k}\|_{\Lambda_T(X)} &= \Big\| \sum_{j \in J_k} \log^{1/2}\left(\frac{2}{2^{n_k} t - j + 1}\right) \cdot \chi_{\Delta_j^{n_k}} \Big\|_X \\
&= \Big\| \log^{1/2}\left(\frac{2\,\mathrm{card}(J_k)}{2^{n_k} t}\right) \cdot \chi_{[0, \frac{\mathrm{card}(J_k)}{2^{n_k}}]} \Big\|_X.
\end{aligned}$$

Since $\operatorname{card}(J_k)\cdot 2^{-n_k}\to m(E)$, the sequence $\log^{1/2}(\frac{2\operatorname{card}(J_k)}{2^{n_k}t})\chi_{[0,\frac{\operatorname{card}(J_k)}{2^{n_k}}]}$ converges pointwise to $\log^{1/2}(\frac{2\,m(E)}{t})\chi_{[0,m(E)]}$. Noting that

$$\log^{1/2}\left(\frac{2\operatorname{card}(J_k)}{2^{n_k}t}\right)\chi_{[0,\frac{\operatorname{card}(J_k)}{2^{n_k}}]}\le\log^{1/2}(2/t)\in X_0,\quad k\in\mathbb{N},$$

and using the separability of X_0, we obtain that

$$\lim_{k\to\infty}\left\|\log^{1/2}\left(\frac{2\operatorname{card}(J_k)}{2^{n_k}t}\right)\chi_{[0,\frac{\operatorname{card}(J_k)}{2^{n_k}}]}\right\|_X=\left\|\log^{1/2}\left(\frac{2\,m(E)}{t}\right)\chi_{[0,m(E)]}\right\|_X.$$

Since $\lim\|\chi_{E_k}\|_{\Lambda_T(X)}=\|\chi_E\|_{\Lambda_T(X)}$, equality (3.7) holds. □

Lemma 3.3 *Let X be an r.i. space on $[0,1]$ such that $\log^{1/2}(2/t)\in X_0$. Then $L_\infty\subset\Lambda_T(X)^\circ$. Moreover, the $\Lambda_T(X)$-norm is rearrangement invariant on L_∞, i.e., $\|g\|_{\Lambda_T(X)}=\|f\|_{\Lambda_T(X)}$ for every equimeasurable functions $f,g\in L_\infty$.*

Proof At first, suppose that f is a step function, that is,

$$f=\sum_{j=1}^m\alpha_j\chi_{E_j},$$

where E_j, $j=1,\dots,m$, are disjoint measurable sets. Since $\Lambda_T(X)^\circ$ is a linear space, from Lemma 3.2, it follows that $f\in\Lambda_T(X)^\circ$. We will prove that

$$\|f\|_{\Lambda_T(X)}=\left\|\sum_{j=1}^m\alpha_j\log^{1/2}\left(\frac{2m(E_j)}{t-\beta_{j-1}}\right)\chi_{(\beta_{j-1},\beta_j]}\right\|_X,\tag{3.8}$$

where $\beta_j:=\sum_{s=1}^j m(E_s)$, with $\beta_0=0$.

For each $j=1,\dots,m$, let $\{E_j^k\}_{k=1}^\infty$ be a sequence of sets such that $\chi_{E_j^k}\in\mathcal{D}$ and $\chi_{E_j^k}\to\chi_{E_j}$ in $\Lambda_T(X)$ as in Lemma 3.2. For a fixed $k\in\mathbb{N}$, the sets $E_1^k,E_2^k,\dots,E_m^k$ need not be disjoint. Let

$$D_k:=\bigcup_{1\le j_1<j_2\le m}E_{j_1}^k\cap E_{j_2}^k.$$

Since the sets E_j, $j=1,\dots,m$, are disjoint and $m(E_j\triangle E_j^k)\to0$ $(k\to\infty)$ for every $j=1,\dots,m$, we have $m(D_k)\to0$.

We write

$$\sum_{j=1}^m\alpha_j\chi_{E_j^k}=\sum_{j=1}^m\alpha_j\chi_{E_j^k\setminus D_k}+\sum_{j=1}^m\alpha_j\chi_{E_j^k\cap D_k}.$$

Note that the sets $E_j^k\setminus D_k$ are disjoint and the functions $h_k:=\sum_{j=1}^m\alpha_j\chi_{E_j^k\setminus D_k}\in\mathcal{D}$. Therefore, by Lemma 3.1,

$$\|h_k\|_{\Lambda_T(X)}=\left\|\sum_{j=1}^m\alpha_j\log^{1/2}\left(\frac{2m(E_j^k\setminus D_k)}{t-\beta_j^{k-1}}\right)\chi_{(\beta_{j-1}^k,\beta_j^k]}\right\|_X,$$

where $\beta_j^k:=\sum_{s=1}^j m(E_s^k\setminus D_k)$ and $\beta_0^k=0$. Since $m(D_k)\to0$ and $\log^{1/2}(2/t)\in X_0$, from this formula, it follows that

$$\lim_{k\to\infty}\|h_k\|_{\Lambda_T(X)}=\left\|\sum_{j=1}^m\alpha_j\log^{1/2}\left(\frac{2m(E_j)}{t-\beta_{j-1}}\right)\chi_{(\beta_{j-1},\beta_j]}\right\|_X,$$

for $\beta_j = \sum_1^j m(E_s)$, with $\beta_0 = 0$. Formula (3.8) is then established noting that, from (3.7), we have

$$\Big\| \sum_{j=1}^{m} \alpha_j \chi_{E_j^k \cap D_k} \Big\|_{\Lambda_T(X)} \le \sum_{j=1}^{m} |\alpha_j| \cdot \|\chi_{E_j^k \cap D_k}\|_{\Lambda_T(X)}$$
$$\le \sum_{j=1}^{m} |\alpha_j| \cdot \Big\| \log^{1/2}\left(\frac{2m(D_k)}{t}\right) \chi_{[0,m(D_k)]} \Big\|_X,$$

which goes to zero as $k \to \infty$ because of $m(D_k) \to 0$ and $\log^{1/2}(2/t) \in X_0$.

Now, let $f \in L^\infty$ and $\{f_k\}$ be a sequence of step functions converging uniformly to f. We can assume that this sequence,

$$f_k = \sum_{j=1}^{m_k} \alpha_j^k \chi_{F_j^k},$$

satisfies the following condition: for every $j \in \{1, \ldots, m_{k+1}\}$, there exists $i \in \{1, \ldots, m_k\}$ with $F_j^{k+1} \subset F_i^k$. Then, by (3.8), we have, for $k > k'$,

$$\|f_k - f_{k'}\|_{\Lambda_T(X)} \le C \cdot \|f_k - f_{k'}\|_\infty \Big\| \sum_{j=1}^{m_k} \log^{1/2}\left(\frac{2m(F_j^k)}{t - \beta_{j-1}^k}\right) \chi_{(\beta_{j-1}^k, \beta_j^k]} \Big\|_X$$
$$= C \cdot \|f_k - f_{k'}\|_\infty \big\| \log^{1/2}(2/t) \big\|_X.$$

Hence, $\{f_k\}$ is a Cauchy sequence with respect to the $\Lambda_T(X)$-norm. So, $f \in \Lambda_T(X)^\circ$ and

$$\|f\|_{\Lambda_T(X)} = \lim_{k\to\infty} \|f_k\|_{\Lambda_T(X)}.$$

Let g be a function equimeasurable with f. Obviously, there is a sequence of step functions $\{g_k\}$ such that g_k is equimeasurable with f_k, for every $k \in \mathbb{N}$, which converges to g uniformly. Noting that, by (3.8), $\|g_k\|_{\Lambda_T(X)} = \|f_k\|_{\Lambda_T(X)}$, $k \in \mathbb{N}$, we obtain $\|g\|_{\Lambda_T(X)} = \|f\|_{\Lambda_T(X)}$. □

Remark 3.4 Let X be an r.i. space on [0, 1] such that $\log^{1/2}(2/t) \in X_0$. Using the formula

$$\|f\|_{\mathrm{Sym}\,(X)} \asymp \|f^*(t) \log^{1/2}(2/t)\|_{X''},$$

proved in [4, Corollary 3.2] and arguing in the same way as in the proofs of Lemmas 3.2 and 3.3, it can be proved that the set $\mathcal{D}$ is dense in L^∞ with respect to the Sym (X)-norm as well.

For every r.i. space X on [0, 1] such that $\log^{1/2}(2/t) \in X_0$, we can define

$$\Lambda_T(X) := \left\{ f \in X : \lim_{n\to\infty} \|f \cdot \chi_{\{t:|f(t)|\le n\}}\|_{\Lambda_T(X)} < \infty \right\},$$

with the norm

$$\|f\|_{\Lambda_T(X)} := \lim_{n\to\infty} \|f \cdot \chi_{\{t:|f(t)|\le n\}}\|_{\Lambda_T(X)}. \tag{3.9}$$

It is clear that $\Lambda_T(X)^\circ$ is a closed subspace of the space $\Lambda_T(X)$, precisely the closure of L_∞ in $\Lambda_T(X)$, that is, $\Lambda_T(X)^\circ = \Lambda_T(X)_0$.

Proposition 3.5 *Let X be an r.i. space on* [0, 1] *with order subcontinuous norm such that $\log^{1/2}(2/t) \in X_0$. The space $\Lambda_T(X)$ is rearrangement invariant and the continuous embeddings $\Lambda_T(X)^\circ \subseteq X_0$ and $\Lambda(X) \subseteq \Lambda_T(X) \subseteq X$ hold.*

Proof Standard arguments show that $\Lambda_T(X)$ is a Banach function lattice. Moreover, by Lemma 3.3, the $\Lambda_T(X)$-norm is r.i. on L_∞-functions. Then, from the definition, it follows that the space $\Lambda_T(X)$ is r.i.

By (3.3), for $f \in \mathcal{D}$, we have $\|f\|_{\Lambda_T(X)} \le C\|f\|_{\Lambda(X)}$. Taking into account Remark 3.4, for every $f \in L_\infty$, we can find a sequence $\{f_k\} \subset \mathcal{D}$ such that $\|f_k - f\|_{\mathrm{Sym}\,(X)} \to 0$. Since $\mathrm{Sym}\,(X) \subset \Lambda(X)$, we have both $\|f_k - f\|_{\Lambda(X)} \to 0$ and $\|f_k - f\|_{\Lambda_T(X)} \to 0$. Hence, $\|f\|_{\Lambda_T(X)} \le C\|f\|_{\Lambda(X)}$ for all $f \in L_\infty$. Since $\Lambda(X)$ is a Banach lattice, then using the definition of the $\Lambda_T(X)$-norm, the last inequality can be extended to the whole space $\Lambda(X)$. Therefore, $\Lambda(X) \subseteq \Lambda_T(X)$.

From (3.6), it follows that $\|f\|_X \le \|f\|_{\Lambda_T(X)}$ for $f \in \mathcal{D}$, which implies $\Lambda_T(X)^\circ \subseteq X_0$. Since X has an order subcontinuous norm, then the continuous embedding $\Lambda_T(X) \subseteq X$ follows from the definition of the space $\Lambda_T(X)$ and (3.9). □

Now, we prove the main result of this section.

Theorem 3.6 *Let X be an r.i. space on $[0,1]$ such that $\log^{1/2}(2/t) \in X_0$. Suppose the Rademacher multiplicator space $\Lambda(X)$ satisfies the following property: if $f \in \Lambda(X)$ and g is a function equimeasurable with f, then $g \in \Lambda(X)$ as well. Then $\Lambda(X) = \Lambda_T(X)$.*

Proof The main step of the proof is showing that there is a constant $C > 0$ such that for every $n \in \mathbb{N}$ and $f \in \mathcal{D}_n$, the following inequality holds

$$\|f\|_{\Lambda(X)} \le C\|f\|_{\Lambda_T(X)}. \tag{3.10}$$

The key idea to do this is to replace f with a more suitable function $f' \in \mathcal{D}_{2n}$ equimeasurable with f and to prove that

$$\|f'\|_{\Lambda(X)} \le C\|f\|_{\Lambda_T(X)}.$$

Then, by hypothesis, the last inequality gives (3.10).

Let $n \in \mathbb{N}$ be arbitrary. Define θ_{ij} to be the value of the function r_j, for $j = 1, \dots, 2n$, on the interval Δ_i^{2n}, with $i = 1, \dots, 2^{2n}$. Given a rearrangement of signs $\varepsilon = (\varepsilon_j)_{j=1}^n \in \{1,-1\}^n$, we define the set

$$\Omega_n^\varepsilon := \left\{ i \in \{1, \dots, 2^{2n}\} : (\theta_{ij})_{j=1}^{2n} \text{ satisfies } \theta_{i,j+n} = \varepsilon_j \cdot \theta_{ij},\ j = 1, \dots, n \right\}. \tag{3.11}$$

We have $\Omega_n^{\varepsilon^1} \cap \Omega_n^{\varepsilon^2} = \emptyset$, whenever $\varepsilon^1 \ne \varepsilon^2$, $\varepsilon^1, \varepsilon^2 \in \{1,-1\}^n$, and

$$\bigcup_{\varepsilon \in \{1,-1\}^n} \Omega_n^\varepsilon = \left\{1, 2, \dots, 2^{2n}\right\}.$$

Associated with each rearrangement of signs $\varepsilon \in \{1,-1\}^n$, we have the set

$$U_n^\varepsilon := \bigcup_{i \in \Omega_n^\varepsilon} \Delta_i^{2n}.$$

Since Ω_n^ε has cardinality 2^n, for every ε, it follows that $m(U_n^\varepsilon) = 2^{-n}$. We also have $U_n^{\varepsilon 1} \cap U_n^{\varepsilon 2} = \emptyset$, whenever $\varepsilon^1 \ne \varepsilon^2$, and

$$\bigcup_{\varepsilon \in \{1,-1\}^n} U_n^\varepsilon = [0,1].$$

Given a function $f \in \mathcal{D}_n$ of the form (3.1) with $c_k \geq 0$ ($k = 1, 2, \ldots, 2^n$) define the function $f' \in \mathcal{D}_{2n}$ as follows:

$$f' = \sum_{k=1}^{2^n} c_k \chi_{U_n^{\varepsilon^k}}, \tag{3.12}$$

where $\{\varepsilon^k\}$ is an enumeration of the set $\{1, -1\}^n$.

By hypothesis, $\Lambda(X)$ coincides with the symmetric kernel Sym (X) of X (see [2, Definition 2.2 and Proposition 2.4]). Therefore, since the functions f and f' are equimeasurable, there exists a constant $A > 0$ independent of f such that

$$\|f\|_{\Lambda(X)} \leq A\,\|f'\|_{\Lambda(X)}. \tag{3.13}$$

The definition of the space $\Lambda(X)$, the condition $\log^{1/2}(2/t) \in X_0$ and a result from [15] mentioned in the Introduction yield immediately that for some $C' > 0$

$$\begin{aligned}\|f'\|_{\Lambda(X)} \leq C' \sup\left\{\left\|f' \cdot \sum_{j=1}^{2n} a_j r_j\right\|_X : \sum_{j=1}^{2n} a_j^2 \leq 1\right\}\\ +C' \sup\left\{\left\|f' \cdot \sum_{j=2n+1}^{\infty} a_j r_j\right\|_X : \sum_{j=2n+1}^{\infty} a_j^2 \leq 1\right\}.\end{aligned} \tag{3.14}$$

It is easy to see that the function $f' \cdot \sum_{j=2n+1}^{\infty} a_j r_j$ is equimeasurable with the function $f \cdot \sum_{j=n+1}^{\infty} a_{j+n} r_j$. Hence, by (2.1), from $\sum_{j=2n+1}^{\infty} a_j^2 \leq 1$, it follows, as in the proof of Lemma 3.1, that

$$\left\|f' \cdot \sum_{j=2n+1}^{\infty} a_j r_j\right\|_X \leq 2\gamma \left\|\sum_{k=1}^{2^n} c_k \log^{1/2} \frac{2}{2^n t - k + 1}\, \chi_{\Delta_k^n}\right\|_X = 2\gamma\,\|f\|_{\Lambda_T(X)},$$

whence

$$\sup\left\{\left\|f' \cdot \sum_{j=2n+1}^{\infty} a_j r_j\right\|_X : \sum_{j=2n+1}^{\infty} a_j^2 \leq 1\right\} \leq 2\gamma\,\|f\|_{\Lambda_T(X)}. \tag{3.15}$$

We now prove a similar inequality for the first supremum from the right-hand side of inequality (3.14).

Due to the fact that θ_{ij} is the value of the function r_j on the interval Δ_i^{2n}, we have

$$g := f' \cdot \sum_{j=1}^{2n} a_j r_j = \sum_{k=1}^{2^n} c_k \sum_{i \in \Omega_n^{\varepsilon^k}} b_i \chi_{\Delta_i^{2n}}, \tag{3.16}$$

where, for $i = 1, \ldots, 2^n$,

$$b_i = \sum_{j=1}^{2n} a_j \theta_{ij}.$$

Since $\|(a_j)\|_2 \leq 1$, then, for $i = 1, \ldots, 2^n$,

$$|b_i| \leq \|(a_j)\|_2 \sqrt{2n} \leq \sqrt{2n}. \tag{3.17}$$

Due to the property defining the sets Ω_n^{ε}, (3.11), we have that, for $i \in \Omega_n^{\varepsilon^k}$,

$$b_i = \sum_{j=1}^{2n} a_j\theta_{ij} = \sum_{j=1}^{n} a_j\theta_{ij} + \sum_{j=1}^{n} a_{j+n}\varepsilon_j^k\theta_{ij} = \sum_{j=1}^{n}(a_j + a_{j+n}\varepsilon_j^k)\theta_{ij}.$$

Since $\|(a_j)\|_2 \le 1$, then, for all $k = 1, \dots, 2^n$, we have

$$\left(\sum_{j=1}^{n}(a_j + a_{j+n}\varepsilon_j^k)^2\right)^{1/2} \le 2. \tag{3.18}$$

Consider, for $k = 1, \dots, 2^n$ and $j = 1, \dots, n$, the sets

$$B_n^k(j) := \left\{i \in \Omega_n^{\varepsilon^k} : |b_i| \ge \sqrt{n-j+1}\right\}.$$

In view of (2.1) and (3.18), we obtain

$$\begin{aligned}
\operatorname{card} B_n^k(j) &= \operatorname{card}\left\{i \in \{1, \dots, 2^{2n}\} : \left|\sum_{s=1}^{n}(a_s + a_{s+n}\varepsilon_s^k)\theta_{is}\right| \ge \sqrt{n-j+1}\right\} \\
&= 2^n m\left(\left\{t \in [0,1] : \left|\sum_{s=1}^{n}(a_s + a_{s+n}\varepsilon_s^k)r_s(t)\right| \ge \sqrt{n-j+1}\right\}\right) \\
&\le 2^n m\left(\left\{t \in [0,1] : \log^{1/2}(2/t) \ge \frac{\sqrt{n-j+1}}{2\gamma}\right\}\right) \\
&= 2\cdot 2^n\cdot 2^{-\frac{n-j+1}{4\gamma^2}}.
\end{aligned} \tag{3.19}$$

We extend the definition of these sets by defining $B_n^k(0) := \emptyset$ and $B_n^k(n+1) := \Omega_n^{\varepsilon^k}$. In view of (3.19), for $k = 1, \dots, 2^n$ and $j = 1, \dots, n+1$, we have, for $\delta = 1/(4\gamma^2) \in (0,1)$,

$$\begin{aligned}
m\left(\bigcup_{i \in B_n^k(j)\setminus B_n^k(j-1)} \Delta_i^{2n}\right) &\le m\left(\bigcup_{i \in B_n^k(j)} \Delta_i^{2n}\right) \\
&= 2^{-2n}\operatorname{card} B_n^k(j) \\
&\le 2\cdot 2^{-n}\cdot 2^{-\delta(n-j+1)}.
\end{aligned} \tag{3.20}$$

Consider, for $j = 1, \dots, n+1$, the functions (compare with (3.16))

$$g_j := \sum_{k=1}^{2^n} c_k \sum_{i \in B_n^k(j)\setminus B_n^k(j-1)} b_i \chi_{\Delta_i^{2n}}$$

and

$$h_j := \sum_{k=1}^{2^n} c_k \chi_{E_k^j}, \quad \text{where } E_k^j = \bigcup_{i \in B_n^k(j)\setminus B_n^k(j-1)} \Delta_i^{2n}.$$

Moreover, denote

$$\alpha_j = 2^{-n}2^{-n\delta}(2^\delta - 1)(1 + 2^\delta + \cdots + 2^{(j-1)\delta}) = 2^{-(1+\delta)n}(2^{j\delta} - 1) \quad (j = 1, \dots, n),$$

$\alpha_0 = 0$ and $\alpha_{n+1} = 2^{-n}$, and the functions

$$v_j := \sum_{k=1}^{2^n} c_k \log^{1/2} \frac{2}{2^n t - k + 1} \chi_{[\frac{k-1}{2^n} + \alpha_{j-1}, \frac{k-1}{2^n} + \alpha_j]} \quad (j = 1, 2, \ldots, n).$$

It is easy to see that $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n+1} = 2^{-n}$.

We first estimate the function g_1. From (3.17), it follows

$$g_1 \le \sqrt{2n}\, h_1. \tag{3.21}$$

Moreover, if

$$u_1 := \sum_{k=1}^{2^n} c_k \log^{1/2} \frac{2}{t - (k-1)2^{-n}} \chi_{(\frac{k-1}{2^n}, \frac{k-1}{2^n} + \alpha_1]},$$

then, by (3.20) and definition of α_j, for $0 < t \le 1$,

$$\sqrt{n(1+\delta)}\, h_1^*(t) \le (\sigma_{2(2^\delta - 1)^{-1}} u_1)^*(t), \tag{3.22}$$

where $\sigma_{2(2^\delta - 1)^{-1}}$ is the dilation operator.

Since $0 < t \le \alpha_1 < 2^{-n(\delta+1)}$ implies that $t^{1-\frac{1}{1+\delta}} \ge 2^n t$, we have, for some constant $C_\delta > 0$ depending on δ and for all $0 < t \le \alpha_1$, that

$$\log^{1/2} \frac{2}{t} \le C_\delta \log^{1/2}(2t^{\frac{1}{1+\delta} - 1}) \le C_\delta \log^{1/2} \frac{2}{2^n t},$$

which implies $u_1 \le C_\delta v_1$. Combining this with (3.21) and (3.22), we have, for $C'_\delta := \sqrt{2}(1+\delta)^{-1/2} C_\delta$,

$$g_1^*(t) \le C'_\delta (\sigma_{2(2^\delta - 1)^{-1}} v_1)^*(t). \tag{3.23}$$

Now, consider the case when $j = 2, \ldots, n$. By definition of the sets $B_n^k(j)$, we have

$$b_i \le \sqrt{n - j + 2}, \quad \text{if } i \notin B_n^k(j-1),$$

and therefore

$$g_j \le \sqrt{n - j + 2}\, h_j. \tag{3.24}$$

Moreover, for some constant $m_\delta > 0$,

$$\log^{1/2} \frac{2}{2^n \alpha_j} = \log^{1/2} \frac{2 \cdot 2^{\delta n}}{2^{\delta j} - 1} \ge m_\delta (n - j + 2)^{1/2}.$$

Hence, since the function $\log^{1/2} \frac{2}{2^n t}$ is decreasing on the interval $(0, 2^{-n}]$, $\alpha_j - \alpha_{j-1} = 2^{-n} \cdot 2^{-\delta(n-j+1)}(2^\delta - 1)$, then using (3.24) and (3.20), we obtain that, for $j = 2, \ldots, n$ and $0 \le t \le 1$,

$$g_j^*(t) \le m_\delta^{-1} (\sigma_{2(2^\delta - 1)^{-1}} v_j)^*(t). \tag{3.25}$$

In view of (3.16), we have $g = \sum_{j=1}^{n+1} g_j$, where the functions g_j are disjoint. Since the function v_j are also disjoint, from (3.23) and (3.25), it follows that

$$\left(\sum_{j=1}^{n} g_j \right)^* (t) \le K' (\sigma_{2(2^\delta - 1)^{-1}} v)^*(t) \tag{3.26}$$

where

$$v := \sum_{k=1}^{2^n} c_k \log^{1/2} \frac{2}{2^n t - k + 1} \chi_{\Delta_k^n},$$

and $K' > 0$ depends only on $\delta > 0$.

Finally, if $i \in \Omega_n^{\varepsilon^k} \setminus B_n^k(n)$, we have that $b_i \leq 1$, which implies $g_{n+1} \leq h_{n+1}$. Furthermore, it is clear that $h_{n+1}^*(t) \leq v^*(t)$ $(0 < t \leq 1)$. Therefore,

$$g_{n+1}^*(t) \leq v^*(t) \quad (0 < t \leq 1)$$

Combining this with (3.26) and with the fact that $\|\sigma_\tau\|_{X \to X} \leq \max(1, \tau)$ in every r.i. space [12, Corollary 2.4.1], we obtain that, for some $K > 0$ depending only on $\delta > 0$ and all (a_j) with $\|(a_j)\|_2 \leq 1$, the following holds

$$\left\| f' \cdot \sum_{j=1}^{2n} a_j r_j \right\|_X = \|g\|_X \leq K \|v\|_X = K \|f\|_{\Lambda_T(X)},$$

whence

$$\sup \left\{ \left\| f' \cdot \sum_{j=1}^{2n} a_j r_j \right\|_X : \sum_{j=1}^{2n} a_j^2 \leq 1 \right\} \leq K \|f\|_{\Lambda_T(X)}.$$

This, together with (3.15), (3.14) and (3.13), implies that, for all $f \in \mathcal{D}$, we obtain inequality (3.10) with $C := C' A (2\gamma + K)$.

By Lemma 3.3, inequality (3.10) extends to all $f \in L_\infty$. Moreover, since $\Lambda(X)$ is r.i., then $\Lambda(X) = \mathrm{Sym}\,(X)$ and therefore it has the Fatou property [4, Proposition 3.1]. Combining this observation with (3.9), we infer $\Lambda_T(X) \subseteq \Lambda(X)$. Since always $\Lambda(X) \subseteq \Lambda_T(X)$ (see Proposition 3.5), the proof is complete. □

Remark 3.7 In the case when $\log^{1/2}(2/t) \in X_0$, by Lemma 3.2, we obtain the following expression for the fundamental function of $\Lambda_T(X)$:

$$\varphi_{\Lambda_T(X)}(u) = \left\| \log^{1/2} \left(\frac{2u}{t} \right) \cdot \chi_{[0,u]}(t) \right\|_X, \quad 0 < u \leq 1.$$

4 Properties of the space $\Lambda_T(X)$

Let X be an r.i. space on $[0, 1]$ such that $\log^{1/2}(2/t) \in X_0$. Proposition 3.5, together with the definition and properties of the Rademacher multiplicator space $\Lambda(X)$ and of its symmetric kernel $\mathrm{Sym}\,(X)$, implies that the following chain of inclusions holds

$$\mathrm{Sym}\,(X) \subset \Lambda(X) \subset \Lambda_T(X) \subset X. \tag{4.1}$$

In this section, we study conditions and consequences of having equality in any of the previous inclusions.

Theorem 4.1 *Let X be an r.i. space on* $[0, 1]$ *such that* $\log^{1/2}(2/t) \in X_0$. *The following conditions are equivalent:*

(i) $\Lambda(X)$ *has the property: if* $f \in \Lambda(X)$ *and* g *is a function equimeasurable with* f, *then* $g \in \Lambda(X)$;

(ii) $\Lambda(X) = \mathrm{Sym}\,(X)$;

(iii) $\Lambda(X) = \Lambda_T(X)$.

Moreover, if X has the Fatou property either from these condition is equivalent to the following

(iv) For an arbitrary function $f \in \mathcal{D}_n$, $n \in \mathbb{N}$, we have

$$\|f\|_{\Lambda(X)} \asymp \sup\left\{\left\|f\sum_{n+1}^{\infty} a_i r_i\|_X : \|\sum_{n+1}^{\infty} a_i r_i\right\|_X \le 1\right\}, \tag{4.2}$$

with constants independent of $n \in \mathbb{N}$ and $f \in \mathcal{D}_n$.

Proof The equivalence of (i), (ii), and (iii) follows from Proposition 3.5, Theorem 3.6 and the fact that, by definition, Sym (X) is the maximal among all r.i. spaces Y such that $Y \subset \Lambda(X)$, [2].

The implication (iii) $\Rightarrow$ (iv) is obvious because of the equivalence of the $\Lambda_T(X)$- and $\Lambda_T(X)^\circ$-norms on the set $\mathcal{D}$. For the reverse implication, let $f \in L^\infty$. By Lemma 3.3, $f \in \Lambda_T(X)^\circ$ and we can choose a sequence $\{f_k\} \subset \mathcal{D}$, which converges to f in the $\Lambda_T(X)$-norm. From (iv), it follows that $\{f_k\}$ is a Cauchy sequence with respect to the $\Lambda(X)$-norm as well. Hence,

$$\|f\|_{\Lambda(X)} = \lim_{k\to\infty} \|f_k\|_{\Lambda(X)} \le C \lim_{k\to\infty} \|f_k\|_{\Lambda_T(X)} = C\|f\|_{\Lambda_T(X)}.$$

If X has the Fatou property, so does $\Lambda(X)$, [2, Proposition 2.6]. Hence, from (3.9), we have that $\Lambda_T(X) \subset \Lambda(X)$. Since the opposite embedding follows from Proposition 3.5, equality (iii) is proved. □

Remark 4.2 In the case when $\log^{1/2}(2/t) \notin X_0$, by [3, Theorem 1] (see also [2, Theorem 3.2]), we have Sym $(X) = \Lambda(X) = L^\infty$. Moreover, taking into account [3, Remark 5], it also follows that (4.2) holds.

Theorem 4.3 *Let X be an r.i. space on* $[0, 1]$ *with order subcontinuous norm such that* $\log^{1/2}(2/t) \in X_0$ *and let α_X be its lower Boyd index. Then*

$$\Lambda_T(X) = X \text{ if and only if } \alpha_X > 0.$$

Proof Let $n \in \mathbb{N}$ and $f \in \mathcal{D}_n$ be defined by (3.1). For each $k = 1, \dots, 2^n$, we have

$$\log^{1/2}\left(\frac{2}{2^n t - k + 1}\right) \cdot \chi_{\Delta_k^n}(t) = \sigma_{2^{-n}}\left(\log^{1/2}(2/\cdot)\right)(t - (k-1)2^{-n}) \cdot \chi_{\Delta_k^n}(t),$$

where $\sigma_{2^{-n}}$ is the dilation operator. Since, for $0 < t \le 1$,

$$\sum_{i=1}^{\infty} \sqrt{i} \cdot \chi_{[\frac{1}{2^i}, \frac{1}{2^{i-1}}]}(t) \le \log^{1/2}(2/t) \le \sum_{i=1}^{\infty} \sqrt{i+1} \cdot \chi_{[\frac{1}{2^i}, \frac{1}{2^{i-1}}]}(t),$$

and, for $n, i \in \mathbb{N}$,

$$\sigma_{2^{-n}} \chi_{[\frac{1}{2^i}, \frac{1}{2^{i-1}}]} = \sigma_{2^{-i}} \chi_{[\frac{1}{2^n}, \frac{1}{2^{n-1}}]},$$

it follows that

$$\sum_{i=1}^{\infty} \sqrt{i} f_i(t) \le \sum_{k=1}^{2^n} c_k \log^{1/2}\left(\frac{2}{2^n t - k + 1}\right) \cdot \chi_{\Delta_k^n}(t) \le \sum_{i=1}^{\infty} \sqrt{i+1} f_i(t), \tag{4.3}$$

where

$$f_i(t) := \sum_{k=1}^{2^n} c_k \cdot \sigma_{2^{-i}} \chi_{[\frac{1}{2^n}, \frac{1}{2^{n-1}}]}(t-(k-1)2^{-n}) \cdot \chi_{\Delta_k^n}(t).$$

Note that the functions f_i are disjoint and, for every $i \in \mathbb{N}$, the function f_i is equimeasurable with the function

$$\sigma_{2^{-i}} f = \sum_{k=1}^{2^n} c_k \sigma_{2^{-i}} \chi_{\Delta_k^n}. \tag{4.4}$$

Suppose now that $\alpha_X > 0$. Then, there exists $\varepsilon > 0$ such that, for all $a \in (0, 1]$, we have $\|\sigma_a f\|_X \le M \cdot a^\varepsilon \|f\|_X$, for some constant $M > 0$ and all $f \in X$. Consequently, from the right-hand side of (4.3), it follows

$$\begin{aligned}
\|f\|_{\Lambda_T(X)} &= \left\| \sum_{k=1}^{2^n} c_k \log^{1/2}\left(\frac{2}{2^n t - k + 1}\right) \cdot \chi_{\Delta_k^n} \right\|_X \\
&\le \sum_{i=1}^{\infty} \sqrt{i+1} \cdot \|\sigma_{2^{-i}} f\|_X \\
&\le M \sum_{i=1}^{\infty} \sqrt{i+1} \cdot 2^{-\varepsilon i} \|f\|_X.
\end{aligned}$$

By Lemma 3.3 and Proposition 3.5, the last inequality extends to the whole space L_∞. Moreover, since X is a Banach lattice, from the definition of the norm of $\Lambda_T(X)$ (see (3.9)), it follows $X \subset \Lambda_T(X)$. Then, Proposition 3.5 implies that $\Lambda_T(X) = X$.

Let, conversely, $\Lambda_T(X) = X$. From the left-hand side of (4.3) and from (4.4), we obtain that, for every $m \in \mathbb{N}$,

$$\begin{aligned}
\|f\|_{\Lambda_T(X)} &= \left\| \sum_{k=1}^{2^n} c_k \log^{1/2}\left(\frac{2}{2^n t - k + 1}\right) \cdot \chi_{\Delta_k^n} \right\|_X \\
&\ge \left\| \sum_{i=1}^{\infty} \sqrt{i} \cdot f_i \right\|_X \\
&\ge \sqrt{m} \cdot \|\sigma_{2^{-m}} f\|_X.
\end{aligned}$$

Consequently, by hypothesis we deduce that, for some constant $C > 0$ and every $m \in \mathbb{N}$,

$$\|\sigma_{2^{-m}} f\|_X \le m^{-1/2} \cdot \|f\|_{\Lambda_T(X)} \le C \cdot m^{-1/2} \cdot \|f\|_X,$$

for every $f \in \mathcal{D}$. Since the norm of X is order subcontinuous, the norm of the dilation operators can be obtained over $\mathcal{D}$. Hence, it follows that

$$\|\sigma_{2^{-m}}\|_{X \to X} \le C \cdot m^{-1/2}, \quad m \in \mathbb{N}.$$

This implies that $\lim_{m\to\infty} \|\sigma_{2^{-m}}\|_{X\to X} = 0$, so $\alpha_X > 0$. □

Consider the inclusions in (4.1). For r.i. spaces X which are "close" to L^∞ (see [3, Theorem 1 and Remark 5]), we have

$$\mathrm{Sym}\,(X) = \Lambda(X) = \Lambda_T(X) \subsetneq X,$$

while in other cases (for example, $X = L^p$ with finite p, see [2, Theorem 2.1] and Theorem 4.3), we have that

$$\Lambda(X) \subsetneqq \Lambda_T(X) = X.$$

Example 4.4 We exhibit an example of an r.i. space X for which the following holds

$$\mathrm{Sym}\,(X) \subsetneqq \Lambda(X) \subsetneqq \Lambda_T(X) \subsetneqq X.$$

Consider the function

$$\psi(t) := 2^{1-\log^{1/2}(2/t)}, \quad 0 < t \le 1.$$

It satisfies, for sufficiently small $t > 0$, the conditions $\psi'(t) > 0$ and $\psi''(t) < 0$. Hence, there exists a nonnegative increasing concave function $\varphi(t)$ on $[0, 1]$ such that $\varphi(t) = \psi(t)$ for sufficiently small $t > 0$. Let $X := M(\varphi)$ be the Marcinkiewicz space with fundamental function φ. It satisfies $\log^{1/2}(2/t) \in X_0$ because of $\lim_{t\to 0^+} \psi(t)\log^{1/2}(2/t) = 0$.

It was proven in [4, Example 5.1] that

$$\|\chi_{[0,2^{-n}]}\|_{\mathrm{Sym}\,(X)} \asymp \|\chi_{[0,2^{-n}]}\|_{\Lambda(X)} \asymp \sqrt{n}\cdot 2^{-\sqrt{n}}, \quad n \in \mathbb{N},$$

but

$$\mathrm{Sym}\,(X) \subsetneqq \Lambda(X).$$

In particular, this implies that the space $\Lambda(X)$ is not r.i. and, therefore, by Proposition 3.5, $\Lambda(X) \subsetneqq \Lambda_T(X)$.

To see that $\Lambda_T(X) \subsetneqq X$, by Theorem 4.3, it is enough to check that $\alpha_X = 0$. An easy calculation shows that $\gamma_\varphi = \gamma_\psi = 0$. Since $\alpha_X = \gamma_\varphi$ [12, § 2.4], required result follows. Another way to prove this is to estimate the norm of $\chi_{[0,2^{-n}]}$ in $\Lambda_T(X)$. By the definition of the $\Lambda_T(X)$-norm and [12, Theorem 2.5.3], we obtain

$$\begin{aligned}\|\chi_{[0,2^{-n}]}\|_{\Lambda_T(X)} &= \left\|\chi_{[0,2^{-n}]}\log^{1/2}\left(\frac{2}{2^n t}\right)\right\|_X \\ &\asymp \sup_{0<t\le 2^{-n}} 2^{1-\log^{1/2}(2/t)}\log^{1/2}\left(\frac{2}{2^n t}\right) \\ &\asymp n^{1/4}\cdot 2^{-\sqrt{n}}, \quad n \in \mathbb{N}.\end{aligned}$$

Since

$$\|\chi_{[0,2^{-n}]}\|_X = \varphi(2^{-n}) \asymp 2^{-\sqrt{n}}, \quad n \in \mathbb{N},$$

from this, it follows that $\Lambda_T(X) \subsetneqq X$.

5 The local version of Khintchine inequality

In this section, we show that a local version of the Khintchine inequality holds for a large class of r.i. spaces X and it is related to the tail Rademacher multiplicator space $\Lambda_T(X)$.

Recall from the Introduction that an r.i. space X with fundamental function φ_X satisfies the *local version of Khintchine inequality* if there exist constants $\alpha, \beta > 0$ such that for any measurable set $E \subset [0, 1]$ with $m(E) > 0$, there exists $N := N(E)$ such that

$$\alpha\varphi_X(m(E))\left\|\sum_{i=N}^{\infty} a_i r_i\right\|_X \le \left\|\sum_{i=N}^{\infty} a_i r_i \chi_E\right\|_X \le \beta\varphi_X(m(E))\left\|\sum_{i=N}^{\infty} a_i r_i\right\|_X \tag{5.1}$$

for every $(a_i) \in \ell^2$ such that $\sum_{i=N}^{\infty} a_i r_i \in X$. In the case when $\log^{1/2}(2/t) \in X_0$, we have $\mathcal{R}(X) \approx \ell^2$ and, therefore, inequality (5.1) is equivalent to

$$\alpha\varphi_X(m(E))\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \leq \left\|\sum_{i=N}^{\infty} a_i r_i \chi_E\right\|_X \leq \beta\varphi_X(m(E))\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2}, \tag{5.2}$$

for every $(a_i) \in \ell^2$.

Since $X \subset L_1$ for every r.i. space X on $[0, 1]$, [12, Theorem 2.41], by Khintchine inequality

$$\left\|\sum_{i=1}^{\infty} a_i r_i\right\|_X \geq c\left(\sum_{i=1}^{\infty} a_i^2\right)^{1/2}, \tag{5.3}$$

for every $(a_i) \in \ell^2$. Therefore, in the general case, the right-hand side inequality from (5.1) is an immediate consequence of the right-hand side inequality from (5.2).

It is noteworthy that the left-hand side of (5.2) holds for any r.i. space X.

Proposition 5.1 *Let X be an r.i. space with fundamental function φ_X. Then, there exists a constant $\alpha > 0$ such that for any measurable set $E \subset [0, 1]$ with $m(E) > 0$ there exists $N := N(E)$ such that*

$$\left\|\sum_{i=N}^{\infty} a_i r_i \chi_E\right\|_X \geq \alpha \cdot \varphi_X(m(E))\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2}$$

for every $(a_i) \in \ell^2$ such that $\sum_{i=N}^{\infty} a_i r_i \in X$.

Proof By a result of Burkholder (see [7], also [18]), there exist universal constants $\eta > 0$ and $\delta > 0$ such that for every set $E \subset [0, 1]$ with $m(E) > 0$, there exists $N := N(E)$ with the following property:

$$m\left(\left\{t \in E : \left|\sum_{i=N}^{\infty} a_i r_i(t)\right| \geq \delta\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2}\right\}\right) \geq \eta \cdot m(E),$$

for any $(a_i) \in \ell^2$. Therefore, if

$$F := \left\{t \in E : \left|\sum_{i=N}^{\infty} a_i r_i(t)\right| \geq \delta\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2}\right\},$$

then, since φ_X is a quasi-concave function and $\eta \leq 1$,

$$\begin{aligned}
\left\|\sum_{i=N}^{\infty} a_i r_i \chi_E\right\|_X &\geq \left\|\sum_{i=N}^{\infty} a_i r_i \chi_F\right\|_X \\
&\geq \delta\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \varphi_X(m(F)) \\
&\geq \delta\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \varphi_X(\eta \cdot m(E)) \\
&\geq \delta\eta\left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \varphi_X(m(E)).
\end{aligned}$$

□

Theorem 5.2 *Let X be an r.i. space on $[0,1]$ with $X \neq L_\infty$. Let γ_{φ_X} be the lower dilation index of the fundamental function φ_X of X. The following conditions are equivalent:*

(i) The local version of Khintchine inequality (5.1) holds;
(ii) Inequality (5.2) holds;
(iii) $\gamma_{\varphi_X} > 0$;
(iv) The function $\log^{1/2}(2/t) \in X_0$ and $\varphi_{\Lambda_T(X)} \asymp \varphi_X$.

Proof (i) ⇔ (ii). To prove this equivalence, it suffices to check that either of conditions (i) or (ii) implies that $\log^{1/2}(2/t) \in X_0$. At first, assume that (i) holds. Let $l \in \mathbb{N}$ be arbitrary. By (5.1), there is $N \geq l$ such that for every $n \in \mathbb{N}$ we have

$$\Big\| \sum_{i=N+1}^{N+n} r_i \chi_{[0,2^{-l}]} \Big\|_X \leq \beta \varphi_X(2^{-l}) \Big\| \sum_{i=N+1}^{N+n} r_i \Big\|_X.$$

Suppose by way of contradiction that $\log^{1/2}(2/t) \notin X_0$. Applying (3.4) with $m = N$, we can find n_0 such that for all $n \geq n_0$

$$\Big\| \sum_{i=N+1}^{N+n} r_i \chi_{[0,2^{-N}]} \Big\|_X \geq c_0 \Big\| \sum_{i=N+1}^{N+n} r_i \Big\|_X.$$

From the above inequalities and the fact that $N \geq l$, we have that $\varphi_X(2^{-l}) \geq c_0/\beta$, for all $l \in \mathbb{N}$. Since the function φ_X is quasi-concave, it follows that $\varphi_X(u) \geq c_0/\beta$, for all $0 < u \leq 1$. Therefore, $X = L_\infty$, which contradicts the assumptions. Moreover, since the right-hand side inequality of (5.2) implies the right-hand side inequality of (5.1), completely similar arguments show that the condition $\log^{1/2}(2/t) \in X_0$ is a consequence of (ii) as well. Thus, equivalence (i) ⇔ (ii) is proved.

(ii) ⇔ (iv). In either case, we can assume that $\log^{1/2}(2/t) \in X_0$. From the definition of the space $\Lambda_T(X)$ and from the fact that the $\|\cdot\|_{\Lambda_T(X)}$–norm is rearrangement invariant on L_∞ (see Lemma 3.3), it follows that the right-hand side of (5.2) holds if and only if there exists a $C_2 > 0$ such that $\varphi_{\Lambda_T(X)}(u) \leq C_2 \varphi_X(u)$, for $0 \leq u \leq 1$. On the other hand, by Proposition 3.5, we have $\varphi_X(u) \leq C_1 \varphi_{\Lambda_T(X)}(u)$, for all $0 \leq u \leq 1$ and some constant $C_1 > 0$. This establishes the equivalence since, by Proposition 5.1, the left-hand side of (5.2) always holds.

(iv) ⇒ (iii). The fundamental function of $\Lambda_T(X)$, by Remark 3.7, is

$$\varphi_{\Lambda_T(X)}(u) = \Big\| \log^{1/2}\Big(\frac{2u}{t}\Big) \cdot \chi_{[0,u]}(t) \Big\|_X, \quad 0 < u \leq 1.$$

From (iv) it follows, for some constant $\beta > 0$, that

$$\Big\| \log^{1/2}\Big(\frac{2u}{t}\Big) \cdot \chi_{[0,u]}(t) \Big\|_X \leq \beta \cdot \varphi_X(u).$$

In particular,

$$\Big\| \log^{1/2}\Big(\frac{2}{2^n t}\Big) \cdot \chi_{[0,2^{-n}]}(t) \Big\|_X \leq \beta \cdot \varphi_X(2^{-n}), \quad n \in \mathbb{N},$$

whence

$$\sqrt{k} \cdot \varphi_X(2^{-n-k}) \leq \beta \cdot \varphi_X(2^{-n}), \quad n, k \in \mathbb{N}.$$

Then,

$$\mathcal{M}_{\varphi_X}(2^{-k}) = \sup_{0<s\leq 1} \frac{\varphi_X(2^{-k}s)}{\varphi_X(s)} \leq 2\frac{\beta}{\sqrt{k}}, \quad k \in \mathbb{N}.$$

Thus, $\mathcal{M}_{\varphi_X}(t) \to 0$ as $t \to 0$. This implies that $\gamma_{\varphi_X} > 0$.

For proving the implication (iii) $\Rightarrow$ (ii), we will need an auxiliary result. $\square$

Lemma 5.3 *Let X be an r.i. space on $[0, 1]$ with the fundamental function φ_X. Suppose that $\gamma_{\varphi_X} > 0$. Then there exists a constant $\beta > 0$ such that, for all $u \in (0, 1]$,*

$$\left\| \log^{1/2}\left(\frac{2u}{t}\right) \chi_{[0,u]} \right\|_X \leq \beta \varphi_X(u).$$

Proof The condition $\gamma_{\varphi_X} > 0$ implies that there exist $\varepsilon > 0$ and $\beta' > 0$ such that

$$\mathcal{M}_{\varphi_X}(t) = \sup_{0<s\leq 1} \frac{\varphi_X(st)}{\varphi_X(s)} \leq \beta' t^\varepsilon, \quad 0 < t \leq 1.$$

In order to prove the result, it suffices to consider the case $u = 2^{-n}$, for $n \in \mathbb{N}$. Then

$$\begin{aligned}
\left\| \log^{1/2}\left(\frac{2}{2^n t}\right) \chi_{[0,2^{-n}]} \right\|_X &\leq \left\| \sum_{k=n}^{\infty} \sqrt{k-n+2}\, \chi_{[2^{-k-1},2^{-k}]} \right\|_X \\
&\leq \sum_{k=n}^{\infty} \sqrt{k-n+2}\, \varphi_X(2^{-k}) \\
&\leq \sum_{k=n}^{\infty} \sqrt{k-n+2}\, \mathcal{M}_{\varphi_X}(2^{-k+n})\, \varphi_X(2^{-n}) \\
&= \sum_{j=0}^{\infty} \sqrt{j+2}\, \mathcal{M}_{\varphi_X}(2^{-j})\, \varphi_X(2^{-n}).
\end{aligned}$$

Consequently,

$$\left\| \log^{1/2}\left(\frac{2}{2^n t}\right) \chi_{[0,2^{-n}]} \right\|_X \leq \beta\, \varphi_X(2^{-n}),$$

for

$$\beta := \beta' \sum_{j=0}^{\infty} \sqrt{j+2} \cdot 2^{-j\varepsilon} < \infty.$$

$\square$

(iii) $\Rightarrow$ (ii). Since, by Proposition 5.1, the left-hand side of (5.2) holds in any r.i. space, we only need to prove the right-hand side of this inequality. Arguing in the same way as in the proof of Theorem 1 of [17], it suffices to consider sets of the form

$$E = \bigcup_{j=1}^{\infty} \Delta_{k_j}^{n_j}, \tag{5.4}$$

where $\Delta_{k_j}^{n_j}$ are disjoint dyadic intervals of $[0, 1]$.

From the condition $\gamma_{\varphi_X} > 0$, it follows that there exist $\varepsilon > 0$, $C > 0$ and $k_0 \in \mathbb{N}$ such that

$$\varphi_X(2^{-k}) \leq C 2^{-\varepsilon k}, \quad k \geq k_0.$$

Hence, for a set E as in (5.4), there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{k=m_0}^{\infty} \sqrt{k+2}\, \varphi_X(2^{-k}) \leq \varphi_X(m(E)). \tag{5.5}$$

Let us write $E = E_1 \cup E_2$, where

$$E_1 := \bigcup_{j=1}^{j_0} \Delta_{k_j}^{n_j}, \quad E_2 := \bigcup_{j=j_0+1}^{\infty} \Delta_{k_j}^{n_j}$$

and j_0 is chosen so that $m(E_2) < 2^{-m_0}$.

By [4, Corollary 3.2], for every function $R = \sum_{i=1}^{\infty} a_i r_i \in X$, we have

$$\begin{aligned}\frac{\|R \cdot \chi_{E_2}\|_X}{\|R\|_X} &\le C_1 \, \|\chi_{E_2}\|_{\Lambda(X)} \\ &\le C_1 \, \|\chi_{E_2}\|_{\mathrm{Sym}\,(X)} \\ &\asymp \left\| \log^{1/2}(2/t)\chi_{[0,m(E_2)]}\right\|_{X''} \\ &\le \left\| \log^{1/2}(2/t)\chi_{[0,2^{-m_0}]}\right\|_{X}.\end{aligned}$$

Taking into account (5.5), we obtain that

$$\begin{aligned}\left\| \log^{1/2}(2/t)\chi_{[0,2^{-m_0}]}\right\|_X &\le \left\| \sum_{k=m_0}^{\infty} \sqrt{k+2}\, \chi_{[2^{-k-1},2^{-k}]}\right\|_X \\ &\le \sum_{k=m_0}^{\infty} \sqrt{k+2}\, \varphi_X(2^{-k}) \\ &\le \varphi_X(m(E)),\end{aligned}$$

which implies that

$$\|R \cdot \chi_{E_2}\|_X \le C_2 \, \varphi_X(m(E)) \cdot \|R\|_X.$$

Since $\gamma_{\varphi_X} > 0$ implies that $(L_N)_0 \subset X$ (see the proof of Theorem 2.1 on p.177 of [2]), we have $\|R\|_X \asymp \left(\sum_{i=1}^{\infty} a_i^2\right)^{1/2}$.

Therefore, we have

$$\left\| \sum_{i=1}^{\infty} a_i r_i \cdot \chi_{E_2}\right\|_X \le C_3 \left(\sum_{i=1}^{\infty} a_i^2\right)^{1/2} \varphi_X(m(E)). \tag{5.6}$$

We prove a similar estimate for the set $E_1 = \bigcup_{j=1}^{j_0} \Delta_{k_j}^{n_j}$. Set $N := \max\{n_j : j = 1, \ldots, j_0\} + 1$. Then, for some set $I_{N-1} \subset \{1, 2, \ldots, 2^{N-1}\}$, we have

$$E_1 = \bigcup_{k \in I_{N-1}} \Delta_k^{N-1}.$$

It is clear that, for every function $R_N = \sum_{i=N}^{\infty} a_i r_i \in X$, the following equality holds

$$\|R_N \cdot \chi_{E_1}\|_X = \left\| \sum_{k \in I_{N-1}} \chi_{\Delta_k^{N-1}} \cdot \left(\sum_{i=N}^{\infty} a_i r_i\right)^* (2^{N-1}t - k + 1)\right\|_X.$$

Combining this with inequality (2.1), we obtain

$$\|R_N \cdot \chi_{E_1}\|_X \le \gamma \left\| \sum_{k \in I_{N-1}} \chi_{\Delta_k^{N-1}} \cdot \log^{1/2}\left(\frac{2}{2^{N-1}t - k + 1}\right)\right\|_X \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2}.$$

For $q := \operatorname{card} I_{N-1}$, the functions

$$\sum_{k \in I_{N-1}} \chi_{\Delta_k^{N-1}} \cdot \log^{1/2}\left(\frac{2}{2^{N-1}t - k + 1}\right)$$

and

$$\chi_{[0,q2^{-N+1}]} \cdot \log^{1/2}\left(\frac{2q}{2^{N-1}t}\right)$$

are equimeasurable. Hence,

$$\|R_N \cdot \chi_{E_1}\|_X \leq \gamma \left\|\chi_{[0,q2^{-N+1}]} \cdot \log^{1/2}\left(\frac{2q}{2^{N-1}t}\right)\right\|_X \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2}.$$

Thus, by Lemma 5.3, we have that

$$\|R_N \cdot \chi_{E_1}\|_X \leq \gamma\beta\, \varphi_X(q2^{-N+1}) \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \leq \gamma\beta\, \varphi_X(m(E)) \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2}.$$

Combining the last inequality with (5.6), we obtain

$$\|R_N \cdot \chi_E\|_X \leq \|R_N \cdot \chi_{E_1}\|_X + \|R_N \cdot \chi_{E_2}\|_X \leq (C_3 + \gamma\beta) \cdot \varphi_X(m(E)) \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2},$$

and the proof is completed.

Remark 5.4 For $X = L^\infty$, it is easy to see that the local Khintchine inequality (5.1) holds (while the right-hand side of (5.2) fails).

Example 5.5 Applying Theorem 5.2 allows to obtain the local Khintchine inequality for the most important classes of r.i. spaces.

a) Let X be a Lorentz–Zygmund space $L^{p,q}(\log L)^\alpha$ (see [5] and [6, §4.6]) with either $1 < p < \infty$, $1 \leq q \leq \infty$ and $\alpha \in \mathbb{R}$, or $p = q = 1$ and $\alpha \geq 0$. Then, there exist constants $A, B > 0$, depending on p, q, α, so that for any measurable set $E \subset [0, 1]$ with $m(E) > 0$ there exists $N := N(E)$ such that

$$A\, \|\chi_E\|_{p,q,\alpha} \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \leq \left\|\sum_{i=N}^{\infty} a_i r_i \chi_E\right\|_{p,q,\alpha} \leq B\, \|\chi_E\|_{p,q,\alpha} \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2},$$

for every $(a_i) \in \ell^2$. In particular, for $\alpha = 0$, we have the Lorentz $L^{p,q}$ spaces and the weak–L^p spaces $L^{p,\infty}$, for which $\|\chi_E\|_{p,q,\alpha} = m(E)^{1/p}$. In the case $p = q < \infty$, we recover Zygmund and Sagher and Zhou results: there exist constants $A'_p, B'_p > 0$ so that for any measurable set $E \subset [0, 1]$ with $m(E) > 0$, there exists $N := N(E)$ such that

$$A'_p \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \leq \left(\frac{1}{m(E)} \int_E \left|\sum_{i=N}^{\infty} a_i r_i(t)\right|^p \mathrm{d}t\right)^{1/p} \leq B'_p \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2},$$

for every $(a_i) \in \ell^2$.

b) Let $X = L_M$ be a separable Orlicz space. Then, there exist constants $\alpha, \beta > 0$ so that for any measurable set $E \subset [0,1]$ with $m(E) > 0$, there exists $N := N(E)$ such that

$$\alpha \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \leq M^{-1}\left(\frac{1}{m(E)}\right) \left\|\sum_{i=N}^{\infty} a_i r_i \chi_E\right\|_{L_M} \leq \beta \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2},$$

for every $(a_i) \in \ell^2$. This follows since the fundamental function of L_M is $\varphi_{L_M}(t) = 1/M^{-1}(1/t)$, and the lower dilation index of φ_{L_M} is strictly positive precisely when L_M is separable.

c) Let $X = M(\varphi)$ be a Marcinkiewicz space. Its fundamental function is equal to the function φ. If the lower dilation index of φ is strictly positive, then there exist constants $\alpha, \beta > 0$ so that for any measurable set $E \subset [0,1]$ with $m(E) > 0$, there exists $N := N(E)$ such that

$$\alpha\, \varphi(m(E)) \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \leq \left\|\sum_{i=N}^{\infty} a_i r_i \chi_E\right\|_{M(\varphi)} \leq \beta\, \varphi(m(E)) \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2},$$

for every $(a_i) \in \ell^2$.

d) For $\varphi\colon [0,1] \to [0,+\infty)$ an increasing concave function and $p \in [1,+\infty)$, let X be the Lorentz space $\Lambda^p(\varphi)$, that is, the space of all measurable functions f on [0,1] such that

$$\|f\|_{\Lambda^p(\varphi)} = \left(\int_0^1 (f^*(s))^p\, d\varphi(s)\right)^{1/p} < \infty.$$

The fundamental function of $\Lambda^p(\varphi)$ is $\phi_{\Lambda^p(\varphi)}(t) = \varphi(t)^{1/p}$. If the lower dilation index of φ is strictly positive, then there exist constants $\alpha, \beta > 0$ so that for any measurable set $E \subset [0,1]$ with $m(E) > 0$, there exists $N := N(E)$ such that

$$\alpha\, \varphi(m(E))^{1/p} \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2} \leq \left\|\sum_{i=N}^{\infty} a_i r_i \chi_E\right\|_{\Lambda^p(\varphi)} \leq \beta\, \varphi(m(E))^{1/p} \left(\sum_{i=N}^{\infty} a_i^2\right)^{1/2},$$

for every $(a_i) \in \ell^2$.

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