ON INTERPOLATION OF BILINEAR OPERATORS WITH A REAL METHOD

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This article considers functors of the real method for interpolation of bilinear operators. A description is obtained for them in the case of exponential characteristic functions.

1. Introduction. We will recall certain definitions from the theory of interpolation of linear operators (for further detail, see [1] or [2]).

A triple of Banach spaces (X_0, X_1, X) is said to be an interpolation triple with respect to a triple (Y_0, Y_1, Y) if continuity of a linear operator T from X_i into Y_i implies continuity of T from X into Y. By an interpolation functor (i.f.) we mean a functor F [1] that maps the category of Banach pairs into the category of Banach spaces so that for arbitrary pairs $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$, the triple $(X_0, X_1, F(\vec{X}))$ is an interpolation triple with respect to the triple $(Y_0, Y_1, F(\vec{Y}))$. We use $s\mathbf{R}$ to denote the one-dimensional space $\mathbf{R}$ with norm $\|x\|_{s\mathbf{R}}=s|x|$, and we use $t\mathbf{R}$ to denote the space $\mathbf{R}$ with norm $\|x\|_{t\mathbf{R}}=t|x|$ $(s, t>0)$. Then, for an interpolation functor F,

$$F(s\mathbf{R}, t\mathbf{R})=\psi(s, t)\mathbf{R},$$

where $\psi(s, t)$ – the characteristic function of the i.f. F – is nonnegative, homogeneous, and increases with each argument. As a result, $\psi(s,t)=t\varphi\left(\frac{s}{t}\right)$, where we also say that the function $\phi(u) = \psi(u, 1)$ is a characteristic function corresponding to the i.f.; $\phi(u)$ $(u > 0)$ is nonnegative and quasiconcave, i.e., $\phi(u)$ increases while $\phi(u)/u$ decreases.

We now introduce a fundamental definition. We say that an i.f. F interpolates bilinear operators if, for any Banach pair $\vec{X}=(X_0, X_1)$, $\vec{Y}=(Y_0, Y_1)$, $\vec{Z}=(Z_0, Z_1)$, and any bilinear operator B: $X_i \times Y_i \to Z_i$ $(i = 0, 1)$ we have B: $F(\vec{X}) \times F(\vec{Y}) \to F(\vec{Z})$.

An important example of interpolation functors is the functors of the real method. For a Banach pair (X_0, X_1) and $t > 0$ we define the Peetre $\mathscr{K}$- and $\mathscr{J}$-functionals as follows [3]:

$$\mathscr{K}(t, x; X_0, X_1) = \inf_{\substack{x=x_0+x_1\\ x_i\in X_i}} \{\|x_0\|_{X_0} + t\|x_1\|_{X_1}\} \ (x \in X_0 + X_1),$$

$$\mathscr{J}(t, x; X_0, X_1) = \max\{\|x\|_{X_0}, t\|x\|_{X_1}\} \ (x \in X_0 \cap X_1).$$

Let E be the ideal Banach space (b.i.s.) [4] of two-sided numerical sequences $a = (a_j)_{-\infty}^{\infty}$. We use $(X_0, X_1)_E^{\mathscr{K}}$ to denote the space of all $x \in X_0 + X_1$ for which $\|x\|=\|(\mathscr{K}(2^j, x; X_0, X_1))_j\|_E$; we use $(X_0, X_1)_E^{\mathscr{J}}$ to denote the space of all $x \in X_0 + X_1$ admitting the representation

$$x = \sum_{j=-\infty}^{\infty} u_j, \tag{1}$$

where $u_j \in X_0 \cap X_1$. The norm in $(X_0, X_1)_E^{\mathscr{J}}$ is defined as $\inf\|(\mathscr{J}(2^j u_j; X_0, X_1))_j\|_E$ over all representations (1). For the b.i.s. of sequences G and a function $f \geq 0$, we use G(f) to denote the weighted space whose norm is given by $\|(a_k)\|=\|(a_k f(2^k))\|_G$. If $\ell_\infty(\max(1, 1/t)) \subset E \subset \ell_1(\min(1, 1/t))$, then the mapping $(X_0, X_1) \mapsto (X_0, X_1)_E^{\mathscr{K}}$ (resp. $(X_0, X_1) \mapsto (X_0, X_1)_E^{\mathscr{J}}$) defines an

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i.f.; the set of them is called the real $\mathcal{K}$-method (resp. $\mathcal{J}$-method) of interpolation. If, in particular, the triple $(\ell_1, \ell_1(1/t), E)$ is an interpolation triple relative to $(\ell_\infty, \ell_\infty(1/t), E)$, we say that E is a parameter of the real method and then, for any pair, $(X_0, X_1)_E^{\mathcal{K}} = (X_0, X_1)_E^{\mathcal{J}}$ [5, p. 432].

An important and well-known special case of the construction we have given is the space

$$(X_0, X_1)_{\theta, p} = (X_0, X_1)^{\mathcal{K}}_{l_p(t^{-\theta})} = (X_0, X_1)^{\mathcal{J}}_{l_p(t^{-\theta})} \; (0 < \theta < 1, 1 \leqslant p \leqslant \infty).$$

In [6], Lions and Peetre (see also [1, p. 100]) proved the following theorem: If T is a bilinear operator, T: $X_i \times Y_i \to Z_i$ $(i = 0, 1)$, then $T\colon (X_0, X_1)_{\theta, p_0} \times (Y_0, Y_1)_{\theta, p_1} \to (Z_0, Z_1)_{\theta, p_2}$, where $0 < \theta < 1$ and

$$\frac{1}{p_2} \leqslant \frac{1}{p_0} + \frac{1}{p_1} - 1. \tag{2}$$

Consider the example of $(X_0, X_1) = (Y_0, Y_1) = (Z_0, Z_1) = (l_1, l_1(t))$ with the following convolution operator for sequences $x = (x_i)$ and $y = y(y_i)$:

$$S(x, y) = x * y = \left(\sum_{i=-\infty}^{\infty} x_i y_{k-i}\right)_k.$$

The hypothesis of the Lions–Peetre theorem is satisfied, and since $(l_1, l_1(t))_{\theta, p} = l_p(t^\theta)$, we have $S\colon (X_0, X_1)_{\theta, p_0} \times (Y_0, Y_1)_{\theta, p_1} \to (Z_0, Z_1)_{\theta, p_2} \Leftrightarrow l_{p_0} * l_{p_1} \subset l_{p_2} \Leftrightarrow \frac{1}{p_2} \leqslant \frac{1}{p_0} + \frac{1}{p_1} - 1.$ As a result, this theorem is exact, and among the functors $(\cdot, \cdot)_{\theta, p}$ $(0<\theta<1, 1 \leqslant p \leqslant \infty)$, bilinear operators are interpolated only by the functors $(\cdot, \cdot)_{\theta, 1}$ $(0 < \theta < 1)$. (We should note that under certain conditions on the Banach pairs, relations (2) can be weakened [7].)

In this paper we study i.f. interpolating bilinear operators, within the framework of the general real method. We show that in case of an exponential i.f. $\phi(t) = t^\theta$, the i.f. $(\cdot, \cdot)_{1-\theta, 1}$ is the only one with this property. However, within "bundles" of functors with certain other characteristic functions, the supply is considerably richer.

We will first prove a result that demonstrates the fundamental nature of our example of the convolution operator.

<u>THEOREM 1.</u> For arbitrary real-method parameters E_0, E_1, and E_2, the following two conditions are equivalent:

1) the convolution $E_0 * E_1 \subset E_2$;

2) for arbitrary Banach pairs (X_0, X_1), (Y_0, Y_1), and (Z_0, Z_1) and any bilinear operator T: $X_i \times Y_i \to Z_i$ $(i = 0, 1)$, we have

$$T\colon (X_0, X_1)_{E_0}^{\mathcal{K}} \times (Y_0, Y_1)_{E_1}^{\mathcal{K}} \to (Z_0, Z_1)_{E_2}^{\mathcal{K}}.$$

<u>Proof</u>. Initially, suppose that 2) holds. We choose $(X_0, X_1) = (Y_0, Y_1) = (Z_0, Z_1) = (l_1, l_1(1/t))$ and for T we choose the convolution S. Since S: $X_i \times Y_i \to Z_i$, we obtain 1) from the relations $(l_1, l_1(1/t))^{\mathcal{K}}_{E_i} = (l_1, l_1(1/t))^{\mathcal{J}}_{E_i} = E_i$ [5, p. 424].

The idea for the proof of the converse is actually drawn from [6]. Thus, let T be a bilinear operator, T: $X_i \times Y_i \to Z_i$. We represent $x \in (X_0, X_1)_{E_0}^{\mathcal{K}}$ and $y \in (Y_0, Y_1)_{E_1}^{\mathcal{K}}$ in the form

$$x = x_0 + x_1 \; (x_i \in X_i), \; y = \sum_{k=-\infty}^{\infty} u_k \; (u_k \in Y_0 \cap Y_1)$$

(we can do this because E_1 is a parameter). Then, because T is bilinear, for all $t > 0$,

$$\mathcal{K}(t, T(x, y); Z_0, Z_1) \leqslant \sum_{k=-\infty}^{\infty} \mathcal{K}(t, T(x, u_k); Z_0, Z_1) \leqslant$$

$$\leqslant \sum_{k=-\infty}^{\infty} (\|T(x_0, u_k)\|_{Z_0} + t\|T(x_1, u_k)\|_{Z_1}) \leqslant$$

$$\leqslant \sum_{k=-\infty}^{\infty} (M_0\|x_0\|_{X_0} + 2^{-k}tM_1\|x_1\|_{X_1}) \mathcal{J}(2^k, u_k; Y_0, Y_1),$$

where $M_i = \|T\|_{X_i \times Y_i \to Z_i}$. If we choose an appropriate representation $x = x_0 + x_1$ for each $k \in \mathbf{Z}$, we obtain

$$\mathcal{K}(t, T(x, y); Z_0, Z_1) \leqslant$$
$$\leqslant \max(M_0, M_1) \sum_{k=-\infty}^{\infty} \mathcal{K}(t2^{-k}, x; X_0, X_1) \mathcal{J}(2^k, u_k; Y_0, Y_1).$$

We now apply this inequality with $t = 2^s$ ($s \in \mathbf{Z}$), together with condition 1), to estimate $\|T(x, y)\|$ in the space $(Z_0, Z_1)_{E_2}^{\mathcal{K}}$:

$$\|T(x, y)\| = \|(\mathcal{K}(2^s, T(x, y); Z_0, Z_1))_s\|_{E_2} \leqslant$$
$$\leqslant \max(M_0, M_1)\|a_x * b_y\|_{E_2} \leqslant C \max(M_0, M_1)\|a_x\|_{E_0}\|b_y\|_{E_1},$$

where $a_x = (\mathcal{K}(2^i, x; X_0, X_1))$, $b_y = (\mathcal{J}(2^i, u_i; Y_0, Y_1))$, and the representation $y = \Sigma u_k$ is chosen in a similar manner.

The theorem we have proved reduces the problem of describing functors interpolating bilinear operators to the study of convolutions in spaces of sequences.

For the case of symmetric functional spaces [2], problems of this type have been considered by a number of authors, and we note, for example, the publications [8-11]. Their results, together with Theorem 1, imply continuity of bilinear operators in certain spaces of the real method with parameters that are symmetric spaces of sequences with exponential weights. However, this does not (and, as Theorem 3 below shows, cannot) yield new examples of functors interpolating bilinear operators.

2. Exponential Case. Henceforth we use e_k ($k \in \mathbf{Z}$) to denote a standard unit vector, i.e., sequence $(\ldots, 0, \underbrace{1}_{k}, 0, \ldots)$.

LEMMA 1. Suppose that G is the b.i.s. of two-sided numerical sequences, $\|e_k\|_G = 1$ ($k \in \mathbf{Z}$). Then there exist a $C > 0$ and an increasing sequence of natural numbers $\{N_k\}_{k=1}^{\infty}$ for which

$$\left\|\sum_{j=-2N_k}^{2N_k} e_j\right\|_G \leqslant C \left\|\sum_{j=-N_k}^{N_k} e_j\right\|_G.$$

Proof. Suppose our assertion is false, i.e.,

$$\lim_{n\to\infty} \frac{\left\|\sum_{j=-2n}^{2n} e_j\right\|}{\left\|\sum_{j=-n}^{n} e_j\right\|} = \infty.$$

Then, for $M > 2$, there exists an $N = N(M)$ such that for $n \geq N$,

$$\left\|\sum_{j=-2n}^{2n}\right\| \geqslant M \left\|\sum_{j=-n}^{n} e_j\right\|.$$

Using this inequality with $n = 2^s N$ ($s = 0, 1, \ldots, p-1$), we obtain

$$\left\|\sum_{j=-2^p N}^{2^p N} e_j\right\| \geqslant M \left\|\sum_{j=-2^{p-1}N}^{2^{p-1}N} e_j\right\| \geqslant \ldots \geqslant M^p \left\|\sum_{j=-N}^{N} e_j\right\|$$

for any p = 1, 2, ...

On the other hand, by hypothesis,

$$\left\|\sum_{j=-2^pN}^{2^pN} e_j\right\| \leqslant \sum_{j=-2^pN}^{2^pN} \|e_j\| = 2^{p+1}N+1.$$

Thus, for p = 1, 2, ...,

$$M^p \leqslant M^p\left\|\sum_{j=-N}^{N} e_j\right\| \leqslant 2^{p+2}N+1,$$

which contradicts the inequality $M > 2$.

<u>THEOREM 2</u>. Suppose that G is a b.i.s. of sequences, $\|e_k\|_G=1$ $(k\in\mathbf{Z})$, and consider the convolution operator $S\colon G\times G\to G$.

Then $G = \ell$.

<u>Proof.</u> Since G is a b.i.s. and $\|e_k\|_G=1$, we have $\ell_1\subset G$. We therefore proceed to proving the reverse inclusion.

Let $y = (y_i) \in E$, $y \geq 0$ and is finite, i.e., for all i, $|i| \geq N$, $y_i = 0$. By Lemma 1, there exist a $C_1 > 0$ and an increasing sequence of natural numbers $\{N_k\}_{k=1}^{\infty}$ such that for all k = 1, 2, ... ,

$$\left\|\sum_{j=-2N_k}^{2N_k} e_j\right\|_G \leqslant C_1\left\|\sum_{j=-N_k}^{N_k} e_j\right\|_G. \tag{3}$$

We choose k_0 so that $N_{k_0} \geq N$, and we write $x=\sum_{j=-2N_{k_0}}^{2N_{k_0}} e_j$. Then $S(x, y) = (z_j)_{-\infty}^{\infty}$, $z_j \geq 0$, and for $|j| \leq N_{k_0}$, we have $z_j=\sum_{i=-N}^{N} y_i=\|y\|_{l_1}$. As a result, $S(x,y)\geqslant Z'=\sum_{j=-N_{k_0}}^{N_{k_0}} z_je_j=\|y\|_{l_1}\sum_{j=-N_{k_0}}^{N_{k_0}} e_j$.* Since G is ideal, it follows from our hypothesis that

$$\|y\|_{l_1}\left\|\sum_{j=-N_{k_0}}^{N_{k_0}} e_j\right\|_G \leqslant \|S(x,y)\| \leqslant C_2\|x\|_G\|y\|_G,$$

where $C_2=\|S\|_{G\times G\to G}$. In view of (3), therefore, $\|y\|_{l_1}\leqslant C_1C_2\|y\|_G$.

Now, if $y = y(y_i) \in G$ is arbitrary, then for the "slice" $y^N=(y_i^N)$, $y_i^N=\begin{cases} y_i, & |i|<N\\ 0, & |i|\geqslant N\end{cases}$ in virtue of the above $\|y^N\|_{l_1}\leqslant C_1C_2\|y^N\|_G\leqslant C_1C_2\|y\|_G$, and C_1 and C_2 are independent of N and y. We can therefore pass to the limit as $N\to\infty$ in the inequality, and the theorem is proved.

<u>THEOREM 3</u>. Suppose the c.f. of the functor $(\cdot,\cdot)_E^{\mathscr{K}}$ constructed with parameter E is t^θ $(0 < \theta < 1)$. The following conditions are equivalent:

1) $E = \ell_1(t^{\theta-1})$;

2) the i.f. $(\cdot,\cdot)_E^{\mathscr{K}}$ interpolates bilinear operators.

<u>Proof</u>. The implication 1)$\Longrightarrow$2) follows from Theorem 1. We will prove the converse.

Since the c.f. of $(\cdot,\cdot)_E^{\mathscr{K}}$ is t^θ, we have $l_1(t^{\theta-1})\subset E\subset l_\infty(t^{\theta-1})$ [12, pp. 47-48]. As a result, for $G = E(t^{1-\theta})$ we have $\|e_k\|_G=1$ $(k\in\mathbf{Z})$. If 2) is satisfied, it follows from Theorem 1 that $S\colon E\times E\to E$, so, in view of the multiplicativeness of exponential functions, $S\colon G\times G\to G$. Thus, the hypothesis of Theorem 2 is satisfied for G, so $G = \ell_1$ and $E = \ell_1(t^{\theta-1})$.

*For $a = (a_i)$, $b = (b_i)$, $a \geq b$ means that $a_i \geq b_i$ for all i.

3. General Case. Now, assume that the c.f. of the functor $(\cdot,\cdot)_E^{\mathcal{K}}$ constructed with parameter E is $\phi(t)$. By Theorem 1, this i.f. interpolates bilinear operators only in the case of a convolution operator S: $E \times E \to E$. In other words, there exists a $C > 0$ such that for all $x=(x_i)\in G$, $y=(y_i)\in G$,

$$\left\|\sum_{i=-\infty}^{\infty} \frac{\varphi_k}{\varphi_i\varphi_{k-i}} x_i y_{k-i}\right\|_G \leqslant C\|x\|\|y\|, \tag{4}$$

where $G=E(\bar{\varphi})$, $\bar{\varphi}(t)=t/\varphi(t)$ and $\phi_s = \phi(2^s)$.

Before we state conditions sufficient for satisfaction of (4), we need some definitions. If G and H are b.i.s. of sequences, we use G[H] to denote the space with mixed norm, i.e., the set of all matrices $(a_{i,k})$ such that

$$\|(a_{i,k})\|_{G[H]}=\|\|(a_{i,k})\|_{H(i)}\|_{G(k)}<\infty.$$

We use G' to denote the b.i.s. dual to the b.i.s. G, i.e., the space of all (y_i) for which

$$\|(y_i)\|_{G'} = \sup_{\|(x_i)\|_G\leqslant 1} \sum_{i=-\infty}^{\infty} x_i y_i < \infty.$$

And, finally, on the product $G \times H$ we define the operator R: $R(x, y) = (x_i y_{k-i})_{i,k}$, where $x=(x_i)\in G$, $y=(y_i)\in H$.

THEOREM 4. Each of the following conditions is sufficient for satisfaction of relation (4):

1) R is continuous from $G \times G$ to $G(k)\,[\ell_\infty(i)]$ and

$$A_\infty(\varphi) = \sup_{k\in\mathbf{Z}} \sum_{i=-\infty}^{\infty} \frac{\varphi_k}{\varphi_i\varphi_{k-i}} < \infty; \tag{5}$$

2) R is continuous from $G \times G$ to $G(k)[G(i)]$ and

$$A_G(\varphi) = \sup_{k\in\mathbf{Z}} \left\|\left(\frac{\varphi_k}{\varphi_i\varphi_{k-i}}\right)\right\|_{G'(i)} < \infty. \tag{6}$$

Proof. If

$$z_k = \sum_{i=-\infty}^{\infty} x_i y_{k-i} \frac{\varphi_k}{\varphi_i\varphi_{k-i}},$$

then, for any $k \in \mathbf{Z}$,

$$|z_k| \leqslant \sup_{i\in\mathbf{Z}} |x_i y_{k-i}| \sup_{k\in\mathbf{Z}} \sum_{i=-\infty}^{\infty} \frac{\varphi_k}{\varphi_i\varphi_{k-i}}.$$

from which it follows that $\|(z_k)\|_G \leqslant A_\infty(\varphi)\|R(x, y)\|_G \leqslant A_\infty(\varphi)\|R\|\|x\|_G\|y\|_G$.

The second condition is obtained in exactly the same way from the inequality

$$|z_k| \leqslant \|(x_i y_{k-i})\|_{G(i)} \sup_{k\in\mathbf{Z}} \left\|\left(\frac{\varphi_k}{\varphi_j\varphi_{k-i}}\right)\right\|_{G'(i)}.$$

Remark 1. There exist nonnegative quasiconcave functions on $(0, \infty)$ that satisfy condition (5) (and, a fortiori, (6)). An example is provided by the family of functions

$$\varphi_{a,b}^{c,d}(t) = \begin{cases} t^a \ln^c(M_1 t^{-1}), & 0 < t \leqslant 1, \\ t^b \ln^d(M_2 t), & t > 1, \end{cases}$$

where $0 < a < b < 1$, $c > 1$, $d > 1$, and the numbers $M_1 > e^{c/a}$ and $M_2 > e^{d/(1-b)}$ are selected on the basis of continuity considerations.

We now consider the case $G = \ell_p$ $(1 \leq p < \infty)$.

THEOREM 5. Let $1 \leqslant p \leqslant \infty$, $\frac{1}{p}+\frac{1}{p'}=1$, and

$$A_p(\varphi)=\sup_{k\in\mathbf{Z}}\varphi_k\left[\sum_{i=-\infty}^{\infty}(\varphi_i\varphi_{k-i})^{-p'}\right]^{1/p'}<\infty. \tag{7}$$

Then the i.f. $(\cdot,\cdot)^{\mathscr{K}}_{l_p(1/\tilde{\varphi})}$ interpolates bilinear operators.

Proof. If $\|x\|_p=\|x\|_{l_p}$, then

$$\|R(x,y)\|_p=\|\|(x_iy_{k-i})\|_{p(i)}\|_{p(k)}=\|\|(x_iy_{k-i})\|_{p(k)}\|_{p(i)}=\|x\|_p\|y\|_p,$$

and, in virtue of the above, the theorem is proved.

Remark 2. For p = 1 and p = ∞, condition (7) is sharp. Indeed, if G = ℓ_∞, then setting $x = y = (1)_{-\infty}^{\infty}$ in (4), we obtain $A_\infty(\phi) < \infty$. Note that (8) appears in the study of the bilinear operator

$$B(x,y)=x(s)y(t) \tag{8}$$

in Marcinkiewicz functional spaces [13].

For p = 1, (7) becomes the condition

$$\sup_{i,k\in\mathbf{Z}}\frac{\varphi_k}{\varphi_i\varphi_{k-i}}<\infty, \tag{9}$$

which is obtained if we use the unit vectors e_k instead of x and y in (4). In view of the quasiconcavity of ϕ, condition (9) is equivalent to generalized polymultiplicativity of this function, i.e., the existence of a C > 0 such that for all s, t > 0, $\phi(st) \leq C\phi(s)\phi(t)$.

Now, consider the case $1 < p < \infty$.

THEOREM 6. Assume that the i.f. $(\cdot,\cdot)^{\mathscr{K}}_{l_p(1/\tilde{\varphi})}$, $1<p<\infty$ interpolates bilinear operators, and for any u > 0, the function $\psi(s)=\frac{\varphi(us)}{\varphi(s)}$ is monotonic on $(0, \infty)$.

Then relation (7) is satisfied.

Proof. Let $z=(z_k)$, $z_k=\sum_{i=-\infty}^{\infty}\frac{\varphi_k}{\varphi_i\varphi_{k-i}}x_iy_{k-i}$. By hypothesis, there exists a C > 0 such that for $x=(x_i)\in l_p$, $y=(y_i)\in l_p$,

$$\|z\|_p\leqslant C\|x\|_p\|y\|_p. \tag{10}$$

Assume that $x \geq 0$ and is finite, $x_i = 0$ if $|i| \geq N$. For a fixed $k \in \mathbf{Z}$, we set $y=\sum_{j=-2N}^{2N}e_{j+k}$.

If $-N + k \leq m \leq N + k$, then $z_m=\sum_{i=-\infty}^{\infty}\frac{\varphi_m}{\varphi_i\varphi_{m-i}}x_i$. As a result,

$$\|z\|_p\geqslant\|v\|_p, \tag{11}$$

where $v=\sum_{i=-\infty}^{\infty}\frac{x_i}{\varphi_i}\sum_{j=-N}^{N}\frac{\varphi_{k+j}}{\varphi_{k-i+j}}e_{j+k}$.

Let I_1 denote the set of all $i \in \mathbf{Z}$ such that the function $\psi(s)=\frac{\varphi(2^is)}{\varphi(s)}$ (and, therefore, $\psi(2^{k-i}s)=\frac{\varphi(2^ks)}{\varphi(2^{k-i}s)}$ as well) increases, and let I_2 denote the set of all i for which it decreases. Then

$$v\geqslant\max\left(\sum_{i\in I_1}\frac{x_i\varphi_k}{\varphi_i\varphi_{k-i}}\sum_{j=0}^{N}e_{j+k},\ \sum_{i\in I_2}\frac{x_i\varphi_k}{\varphi_j\varphi_{k-i}}\sum_{j=-N}^{0}e_{j+k}\right)$$

and so

$$\|v\|_p\geqslant\max\left(\sum_{i\in I_1}\frac{x_i\varphi_k}{\varphi_i\varphi_{k-i}},\ \sum_{i\in I_2}\frac{x_i\varphi_k}{\varphi_i\varphi_{k-i}}\right)(N+1)^{1/p}.$$

In view of (10) and (11) it follows that for any finite $x = (x_i)$, $\|x\|_p = 1$,

$$\max\left(\sum_{i\in I_1} \frac{|x_i|\varphi_k}{\varphi_i\varphi_{k-i}}, \sum_{i\in I_2} \frac{|x_i|\varphi_k}{\varphi_i\varphi_{k-i}}\right) \leqslant 2^{1/p}C.$$

Passing to the sharp upper bound with respect to (x_i), we obtain

$$\max\left\{\left[\sum_{i\in I_1}\left(\frac{\varphi_k}{\varphi_i\varphi_{k-i}}\right)^{p'}\right]^{1/p'}, \left[\sum_{i\in I_2}\left(\frac{\varphi_k}{\varphi_i\varphi_{k-i}}\right)^{p'}\right]^{1/p'}\right\} \leqslant 2^{1/p}C,$$

from which it follows, because $I_1 \cup I_2 = \mathbf{Z}$, that

$$\left[\sum_{i=-\infty}^{\infty}\left(\frac{\varphi_k}{\varphi_i\varphi_{k-i}}\right)^{p'}\right]^{1/p'} \leqslant 2^{1+1/p}C.$$

The theorem is proved.

As another example, consider a generalization of ℓ_p — the scale of Lorentz spaces $\ell_{p,q}$ $(1 < p < \infty,\ 1 \leq q \leq \infty)$: A bounded sequence $x = (x_k)_{-\infty}^{\infty} \in l_{p,q}$ if

$$\|x\|_{p,p} = \begin{cases} \left[\sum_{i=1}^{\infty} (x_i^*)^q\, i^{q/p-1}\right]^{1/q}, & q < \infty, \\ \sup\limits_{i\in\mathbf{N}} (x_i^* i^{1/p}), & q = \infty, \end{cases}$$

is finite, where $(x_i^*)_{i=1}^{\infty}$ is a permutation of (x_k) in nonincreasing order [14, p. 146]. In particular, $\ell_{p,p} = \ell_p$.

<u>PROPOSITION</u>. The bilinear operator $R(x, y) = (x_i, y_{k-i})$ is continuous from $\ell_{p,q} \times \ell_{p,q}$ to $l_{p,q}(k)[l_{(p-1)q'}(i)]$ if $1 < p < \infty$ and $1 \leq q \leq p$.

<u>Proof</u>. Initially, assume $q = 1$. We fix $x = (x_k) \in \ell_{p,1}$ and set $T_x y = R(x, y)$. The operator T_x is linear, and we will estimate its values on the characteristic functions $\chi_e(i) = \begin{cases} 1, & i \in e, \\ 0, & i \notin e \end{cases}$. We should note that when $\operatorname{card}(e) = r$, we have

$$\|T_x\chi_e\|_\infty \leqslant S_r x,$$

where $S_r x = (\underbrace{x_1^*, \ldots, x_1^*}_{r}, \ldots, \underbrace{x_k^*, \ldots, x_k^*}_{r}, \ldots)$.

As a result,

$$\begin{aligned} \|\|T_x\chi_e\|_{\infty(i)}\|_{p,1(k)} &\leqslant \|S_r x\|_{p,1} = \\ &= \sum_{k=1}^{\infty} x_k^* \sum_{i=(k-1)r+1}^{kr} i^{1/p-1} \leqslant pr^{1/p} \sum_{k=1}^{\infty} x_k^* (k^{1/p} - (k-1)^{1/p}) \leqslant \\ &\leqslant 2pr^{1/p} \sum_{k=1}^{\infty} x_k^* k^{1/p-1} = 2p\|x\|_{p,1}\|\chi\|_{p,1}. \end{aligned}$$

The discrete form of Lemma 5.2 of Ch. 2 of [2] allows us to conclude that $\|T_x\|$ from $\ell_{p,1}$ to $\ell_{p,1}^{(k)}[\ell_\infty(i)]$ does not exceed $4p\|x\|_{p,1}$. This means that $R\colon l_{p,1}\times l_{p,1} \to l_{p,1}(k)[l_\infty(i)]$ and $\|R\| \leqslant 4p$.

When $1 < q < p$, we use the complex method of interpolation [1, ch. 4]. According to what we have proved,

$$\begin{aligned} &R\colon l_p \times l_p \to l_p(k)[l_p(i)], \\ &R\colon l_{p,1} \times l_{p,1} \to l_{p,1}(k)[l_\infty(i)]. \end{aligned} \tag{12}$$

We choose $\theta: \frac{1-\theta}{1}+\frac{\theta}{p}=\frac{1}{q}$. As we know [1, p. 126 in Russian edition], for each $0 < \theta < 1$, the complex method interpolates bilinear operators. In addition, $[l_{p,1}, l_p]_\theta = l_{p,q}$ (see [14], p. 146 in Russian edition] and [1, p. 134 in Russian edition]), and for any b.i.s. E_i and F_i ($i = 0, 1$), we have $[E_0[F_0], E_1[F_1]]_\theta = [E_0, E_1]_\theta[[F_0, F_1]_\theta]$ [15]. As a result, interpolating (12), we obtain

$$R: l_{p,q}\times l_{p,q}\to l_{p,q}(k)[[l_p(i), l_\infty(i)]_\theta].$$

Since $\theta = p'/q'$, we have $[l_p, l_\infty]_\theta = l_{(p-1)q'}$ [1, p. 139 in Russian edition] and our proposition is proved.

Our proposition and Theorem 4 imply

THEOREM 7. Let $i < p < \infty$, $1 \le q \le p$, and assume that condition (5) is satisfied for the function ϕ.

Then the i.f. $(\cdot, \cdot)^{\mathscr{K}}_{l_{p,q}(1/\tilde{\varphi})}$ interpolates bilinear operators.

Remark 3. We assume that the norm of a b.i.s. of sequences E is order semicontinuous, i.e., $0 \leq x_n \uparrow x$ ($x_n, x \in E$) implies that $\|x_n\|_E \to \|x\|_E$. The continuity of R from $E \times E$ to $E(k)[\ell_\infty(i)]$ is equivalent to continuity of the operator $V(x, y) = (x_k y_i)$ that is the discrete analog of the operator B (see (8)) from $E \times E$ to $E(\mathbf{Z} \times \mathbf{Z})$. This makes it possible to show that $R: l_{p,\infty}\times l_{p,\infty}\not\to l_{p,\infty}(k)[l_\infty(i)]$. The author does not know whether the operator R is continuous from $\ell_{p,1} \times \ell_{p,1}$ to $\ell_{p,1}(k)[\ell_{p,1}(i)]$.

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