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$\Lambda(p)$-spaces ☆

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ABSTRACT

We say that a closed subspace H of the space $L_p = L_p[0,1]$, $1 \leqslant p < \infty$, is a $\Lambda(p)$-*space* if, in H, convergence in L_p-norm is equivalent to convergence in measure. Mainly, we focus on the problem when the unit ball $B_H := \{f \in H\colon \|f\|_p \leqslant 1\}$ of a $\Lambda(p)$-space H has equi-absolutely continuous norms in L_p. Moreover, assuming that a rearrangement invariant space X is embedded into L_p, $1 \leqslant p < \infty$, and that the inclusion operator $I : X \to L_p$ fails to be strictly singular, we are interested in what we can say about the properties of subspaces on which the norms of X and L_p are equivalent. We reveal their essential dependence on the value of p, which resembles the difference in the classical theorems of Bourgain and Bachelis–Ebenstein on $\Lambda(p)$-sets.

1. Introduction

The classical Khintchine inequality (see e.g. [32, Chapter V, Theorem 8.4]) asserts that, for every $0 < p < \infty$, there exist constants $A_p, B_p > 0$ such that for any sequence of real numbers $\{c_k\}_{k=1}^{\infty}$ we have

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$$A_p\|(c_k)\|_2 \leqslant \left\|\sum_{k=1}^{\infty} c_k r_k\right\|_{L_p[0,1]} \leqslant B_p\|(c_k)\|_2, \tag{1}$$

where $\|(c_k)\|_2 := (\sum_{k=1}^{\infty} c_k^2)^{1/2}$ and r_k's are the Rademacher functions, that is, $r_k(t) = \operatorname{sign}(\sin 2^k\pi t)$, $k \in \mathbb{N}$, $t \in [0,1]$. Thus, for every $0 < p < \infty$ the sequence $\{r_k\}$ is equivalent in L_p to the standard basis of l_2. The following notion reflects general phenomena exemplified above by the Rademacher system. We say that a closed subspace H of the space $L_p = L_p[0,1]$, $1 \leqslant p < \infty$, is a $\Lambda(p)$-*space* if, in H, convergence in L_p-norm is equivalent to convergence in measure. Equivalently, for each (for some) $0 < q < p$ there exists a constant $C_q > 0$ such that

$$\|f\|_p \leqslant C_q\|f\|_q \quad \text{for all } f \in H \tag{2}$$

(see e.g. [1, Proposition 6.4.5]).

Inequality (1) shows that the closed linear span $[r_k]$ (say, in L_1) is a $\Lambda(p)$-space for every $1 \leqslant p < \infty$. At the same time, the closed linear span $[\xi_k]$ of a sequence $\{\xi_k\}_{k=1}^{\infty}$ of independent identically distributed s-stable random variables ($1 < s < 2$) in L_1 is a $\Lambda(p)$-space if and only if $p \in (0,s)$ (see e.g. [1, §6.4]). Note that sometimes a different terminology is used; for instance, in the book [1] (see Definition 6.4.4) a subspace H satisfying (2) is said to be strongly embedded in L_p.

It is worth emphasizing that originally the concept of a $\Lambda(p)$-space was introduced by Rudin [28], in the special setting of Fourier analysis on the circle group $[0,2\pi)$. Namely, let $0 < p < \infty$. A set $E \subset \mathbb{Z}$ is called a $\Lambda(p)$-*set* if for some $0 < q < p$ there exists a constant $C_q > 0$ such that inequality (2) holds for every polynomial f with spectrum (i.e., support of Fourier transform) in E. Equivalently, the subspace $H = L_E$ generated by a set of exponentials $\{e^{2\pi inx},\ n \in E\}$ is a $\Lambda(p)$-space. In [28], for all integers $n > 1$ Rudin constructed $\Lambda(2n)$-sets that are not $\Lambda(q)$-sets for every $q > 2n$. In 1989, Bourgain proved a deep result extending Rudin's theorem to all $p > 2$ [7]. Earlier, in 1974, Bachelis and Ebenstein [3] showed that in the case when $p \in (1,2)$ every $\Lambda(p)$-set is a $\Lambda(q)$-set for some $q > p$. Note that the gap between Bourgain theorem and Bachelis–Ebenstein theorem, i.e., the question whether any $\Lambda(2)$-set is a $\Lambda(q)$-set for some $q > 2$ remains still open (the so-called $\Lambda(2)$-set problem).

In this paper, $\Lambda(p)$-spaces are considered in a more general setting of rearrangement invariant (r.i.) spaces. Mainly, we focus on the problem when the unit ball $B_H := \{f \in H\colon \|f\|_p \leqslant 1\}$ of a $\Lambda(p)$-space H has equi-absolutely continuous norms in L_p. Moreover, assuming that an r.i. space X is embedded into L_p, $1 \leqslant p < \infty$, and that the inclusion operator $I : X \to L_p$ fails to be strictly singular, we are interested in what we can say about the properties of subspaces on which the norms of X and L_p are equivalent. This approach allows us to reveal an essential dependence of results on the value of p, which resembles the difference in the above-mentioned theorems of Bourgain and Bachelis–Ebenstein.

Clearly, the notion of a $\Lambda(p)$-space naturally extends to an arbitrary r.i. space. In [30], Tokarev introduced and studied the class $K(X)$ of all infinite dimensional subspaces B of an r.i. space X on the interval $[0,1]$ such that convergence in the norm $\|\cdot\|_X$ is equivalent in B to convergence in measure. Moreover, in [31], he announced (without proofs) a series of results related to the family $T(B)$ (where B is a fixed reflexive subspace of L_1) of all r.i. spaces X such that $B \subset X$ and convergence in the norm $\|\cdot\|_X$ is equivalent in B to convergence in L_1-norm. Here, we do not pursue such a goal and confine ourselves to the case of $\Lambda(p)$-spaces.

The paper is organized as follows. After Section 2, where some necessary definitions and auxiliary results are collected, in Section 3, we prove that every subspace H of L_p, $1 \leqslant p < \infty$, is a $\Lambda(p)$-space whenever there is an r.i. space $X \subset L_p$ such that $H \subset X \subset L_p$ and the inclusion operator $I : X \to L_p$ is disjointly strictly singular (Theorem 1). In the opposite direction, applying the above Bourgain's result on $\Lambda(p)$-spaces spanned by sequences of uniformly bounded orthogonal functions [7], we show that for every $p > 2$ there is a $\Lambda(p)$-space, which is not embedded into any Orlicz space L_N such that $L_N \overset{\neq}{\subset} L_p$ and therefore whose unit ball fails to have equi-absolutely continuous norms in L_p (Example 1). Moreover, we are able to construct a $\Lambda(2)$-space spanned by a sequence of mean zero independent functions, which is not embedded into any r.i. space X such that $X \overset{\neq}{\subset} L_2$ (Example 2).

Section 4 contains main results of the paper. First, in Theorem 2, we prove that, for $1 \leqslant p < 2$, the unit ball of arbitrary $\Lambda(p)$-space has equi-absolutely continuous norms in L_p. It is not the case if $p \geqslant 2$. However, if H is a closed subspace of L_2 such that there exists an r.i. space X with the properties that $H \subset X \subset L_2$ and that the inclusion operator $I : X \to L_2$ is disjointly strictly singular, then the unit ball B_H has equi-absolutely continuous norms in L_2 (Theorem 3). On the contrary, using constructions of the paper [11], for arbitrary $p > 2$ we are able to find an Orlicz space L_F embedded into L_p such that the inclusion operator $I : L_F \to L_p$ is disjointly strictly singular and the norms of L_p and L_F are equivalent on some closed subspace H of L_p whose unit ball B_H fails to have equi-absolutely continuous norms in L_p (Theorem 4).

Finally, in Section 5 we consider uniformizable $\Lambda(2)$-spaces introduced by Blei in the book [6] (see also [5]) and extend to them some results proved earlier in [8] in the special case of $\Lambda(2)$-sets. In particular, we show that a closed subspace H of L_2 is a uniformizable $\Lambda(2)$-space if and only if the unit ball B_H has equi-absolutely continuous norms in L_2 (Theorem 5) and prove that the orthogonal sum of two $\Lambda(2)$-spaces is still a $\Lambda(2)$-space provided that at least one of them is uniformizable (Theorem 7).

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2. Preliminaries

In what follows given two positive functions (quasinorms) f and g are said to be equivalent (we write $f \asymp g$) if there exists a positive constant C such that $C^{-1}f \leqslant g \leqslant Cf$. Throughout we use the letters C and c to denote positive constants whose values may change at different occurrences.

2.1. Rearrangement invariant spaces

Detailed exposition of theory of rearrangement invariant spaces see in [20,22,4].

A Banach space X of real-valued Lebesgue-measurable functions on the interval $[0,1]$ is called *rearrangement invariant* (*r.i.*) if from the conditions $y \in X$ and $x^*(t) \leqslant y^*(t)$ almost everywhere (a.e.) on $[0,1]$ it follows that $x \in X$ and $\|x\|_X \leqslant \|y\|_X$. Here and throughout, m is the Lebesgue measure and $x^*(t)$ is the left-continuous non-increasing *rearrangement* of $|x(s)|$, which is defined by

$$x^*(t) := \inf\bigl\{\tau \geqslant 0\colon\ m\bigl\{s \in [0,1]\colon\ \bigl|x(s)\bigr| > \tau\bigr\} < t\bigr\} \quad (t > 0).$$

In particular, if x and y are *equimeasurable* functions, that is,

$$m\bigl\{s \in [0,1]\colon\ \bigl|x(s)\bigr| > \tau\bigr\} = m\bigl\{s \in [0,1]\colon\ \bigl|y(s)\bigr| > \tau\bigr\} \quad \text{for all } \tau > 0,$$

and $y \in X$ then $x \in X$ and $\|x\|_X = \|y\|_X$.

For each r.i. space X on $[0,1]$ the continuous embeddings $L_\infty \subset X \subset L_1$ hold. By $X_\circ$ we will denote *the separable part* of X, i.e., the closure of L_∞ in X; $X_\circ$ is an r.i. space which is separable provided that $X \neq L_\infty$. Throughout χ_A is the characteristic function of a set A. The *fundamental function* of an r.i. space X is the function $\varphi_X(t) := \|\chi_{[0,t]}\|_X$, $0 \leqslant t \leqslant 1$. The *lower and upper dilation indices* of φ_X are defined by

$$\gamma_{\varphi_X} = \lim_{t\to 0+} \frac{\log \mathbf{M}_{\varphi_X}(t)}{\log t} \quad \text{and} \quad \delta_{\varphi_X} = \lim_{t\to\infty} \frac{\log \mathbf{M}_{\varphi_X}(t)}{\log t},$$

respectively, where $\mathbf{M}_{\varphi_X}(t) = \sup_{0<s\leqslant 1, 0<st\leqslant 1} \varphi_X(st)/\varphi_X(s)$. Since φ_X is *quasi-concave* (i.e., φ_X increases and $\varphi_X(t)/t$ decreases), we have $0 \leqslant \gamma_{\varphi_X} \leqslant \delta_{\varphi_X} \leqslant 1$.

Most important examples of r.i. spaces are the L_p-spaces ($1 \leqslant p \leqslant \infty$) and their natural generalization, the Orlicz spaces [19]. Let $N(u)$ be an Orlicz function on $[0,\infty)$, that is, a continuous convex increasing function on $[0,\infty)$ such that $N(0) = 0$. Then the *Orlicz space* L_N consists of all measurable functions $x(t)$ on $[0,1]$ such that $\int_0^1 N(|x(t)|/\lambda)\,dt \leqslant 1$, for some $\lambda > 0$. The *Luxemburg norm* $\|x\|_{L_N} := \inf \lambda$, where the infimum is taken over all λ satisfying the last inequality. An Orlicz space L_N is separable if and only if the function N satisfies the Δ_2-condition at infinity (i.e., there exist $u_0 > 0$ and $B > 0$ such that $N(2u) \leqslant BN(u)$ for all $u > u_0$). The fundamental function of L_N is $\varphi_{L_N}(s) = 1/N^{-1}(1/s)$, where N^{-1} is the inverse function to N.

2.2. Subsets of r.i. spaces with equi-absolutely continuous norms

Let X be an r.i. space on $[0,1]$. We shall say that a set $K \subset X$ has *equi-absolutely continuous norms* in X if

$$\lim_{\delta\to 0} \sup_{m(E)<\delta} \sup_{f\in K} \|f\chi_E\|_X = 0.$$

Two statements below can be proved by standard arguments (see e.g. [1, Lemmas 5.2.6 and 5.2.7]).

Lemma 1. *For arbitrary r.i. space X on $[0,1]$ such that $X \neq L_\infty$ and for any set $K \subset X$ the following conditions are equivalent*:

(i) *K has equi-absolutely continuous norms in X*;

(ii)

$$\lim_{M\to\infty} \sup_{f\in K} \|f\chi_{\{|f|>M\}}\|_X = 0.$$

Lemma 2. *Let $1 \leqslant p < \infty$, and let H be a closed subspace of $L_p = L_p[0,1]$. If the set $B_H := \{f \in H\colon \|f\|_p \leqslant 1\}$ has equi-absolutely continuous norms in L_p, then H is a $\Lambda(p)$-space.*

2.3. Inclusions of r.i. spaces

A linear operator between two Banach spaces E and F is called *strictly singular*, or *Kato*, if it fails to be an isomorphism on any infinite dimensional subspace of E (cf. [21, 2.c.2]). The class of all strictly singular operators, which was introduced by Tosio Kato in [18], is a well-known closed operator ideal with important applications. A weaker notion for Banach lattices, introduced in [11], is the following one: a bounded operator T from a Banach lattice E to a Banach space F is said to be *disjointly strictly singular* (DSS, in short) if there is no disjoint sequence of non-null vectors $\{x_n\}_{n=1}^\infty$ in E such that the restriction of T on the subspace $[x_n]$ spanned by the vectors x_n $(n = 1, 2, \ldots)$ is an isomorphism. This is a useful tool in studying the structure of r.i. spaces (cf. [11,9]) Clearly, every strictly singular operator is disjointly strictly singular but the converse is not true in general; it is sufficient to consider the canonic inclusion $L_q[0,1] \to L_p[0,1]$, $1 \leqslant p < q < \infty$, and the sequence of Rademacher functions (see Introduction). In general, the class of all disjointly strictly singular operators is not an operator ideal since it fails to be stable with respect to the composition on the right.

Let X and Y be a pair of r.i. spaces on $[0,1]$ and $X \subset Y$. The inclusion $X \subset Y$ is called *strong* if

$$\lim_{\delta\to 0} \sup_{\|f\|_X\leqslant 1, m(\operatorname{supp} f)\leqslant\delta} \|f\|_Y = 0$$

(supp f is the support of a measurable function f). If an inclusion $X \subset Y$ is strong, then $\lim_{t\to 0} \varphi_Y(t)/\varphi_X(t) = 0$, where φ_X and φ_Y are the fundamental functions of the spaces X and Y, respectively. The inverse statement does not hold. More precisely, in [2] it is constructed a pair of r.i. spaces X and Y with $X \subset Y$ such that $\lim_{t\to 0} \varphi_Y(t)/\varphi_X(t) = 0$ and the inclusion $X \subset Y$ is not DSS. On the other hand, obviously, any strong inclusion is DSS. Moreover, there are Orlicz spaces L_G and L_N such that the inclusion $L_G \subset L_N$ is DSS but not strong [11, Proposition 3.2]. We will need the following simple sufficient condition under which the latter inclusion is strong (see also [10]).

Lemma 3. *Suppose G and N are two Orlicz functions such that N satisfies the Δ_2-condition at infinity and $\lim_{u\to\infty} G(u)/N(u) = \infty$. Then $L_G \subset L_N$ and this inclusion is strong.*

Proof. Firstly, the inclusion $L_G \subset L_N$ follows from the assumption and the fact that the norm of any Orlicz space on $[0,1]$ is determined (up to equivalence) by the behavior of the corresponding Orlicz function for large values of argument.

Now, since N satisfies the Δ_2-condition at infinity, for arbitrary $\varepsilon > 0$ there exist $u_0 > 0$ and $B = B_\varepsilon > 0$ such that $N(u/\varepsilon) \leqslant BN(u)$ for all $u > u_0$. Moreover, let $M_0 = M_0(\varepsilon) > u_0$ and $\delta > 0$ be such that $N(u) \leqslant (2B)^{-1}G(u)$ for all $u \geqslant M_0$ and $2\delta N(M_0/\varepsilon) < 1$. Then for arbitrary $f \in L_G$, with $\int_0^1 G(|f(x)|)\,dx \leqslant 1$ and $m(\operatorname{supp} f) \leqslant \delta$, we have

$$\begin{aligned}
\int_0^1 N\left(\frac{|f(x)|}{\varepsilon}\right) dx &= \int_{\{|f(x)|>M_0\}} N\left(\frac{|f(x)|}{\varepsilon}\right) dx + \int_{\{|f(x)|\leqslant M_0\}} N\left(\frac{|f(x)|}{\varepsilon}\right) dx \\
&\leqslant B \int_{\{|f(x)|>M_0\}} N\big(|f(x)|\big)\,dx + N\left(\frac{M_0}{\varepsilon}\right) m(\operatorname{supp} f) \\
&\leqslant \frac{1}{2}\int_0^1 G\big(|f(x)|\big)\,dx + N\left(\frac{M_0}{\varepsilon}\right)\delta \leqslant 1.
\end{aligned}$$

Thus, $\|f\|_{L_N} \leqslant \varepsilon$, and the proof is complete. $\square$

The following lemma is a version of the well-known De La Vallee Poussin theorem on uniformly integrable sets in L_1 (see, for instance, [23, pp. 19–20] and in a more general setting [12, Proposition 6.6, p. 136]) and can be proved in a similar way.

Recall that an Orlicz function N is said to be *p-convex* if the map $t \mapsto N(t^{1/p})$ is convex.

Lemma 4. *Let $1 \leqslant p < \infty$ and let $K \subset L_p = L_p[0,1]$ be an arbitrary set.*

(i) *Suppose that there exists a non-decreasing function $N(u)$ on $[0,\infty)$, $N(0) = 0$, such that*

$$\lim_{u\to\infty} \frac{N(u)}{u^p} = \infty \tag{3}$$

and

$$\sup_{f\in K} \int_0^1 N\big(\big|f(x)\big|\big)\,dx < \infty. \tag{4}$$

Then K has equi-absolutely continuous norms in L_p.

(ii) *If the set K has equi-absolutely continuous norms in L_p, then there exists a non-decreasing function $N(u)$ on $[0,\infty)$, $N(0) = 0$, which is equivalent to some p-convex function and satisfies the Δ_2-condition and relations (3) and (4).*

By Lemma 3, condition (3) guarantees that the inclusion $L_N \subset L_p$ is strong. Therefore, from Lemma 4 it follows

Proposition 1. *Let $1 \leqslant p < \infty$, and let H be a closed subspace of L_p. The following conditions are equivalent*:

(i) *the unit ball of H has equi-absolutely continuous norms in L_p;*
(ii) *there is a p-convex Orlicz function $N(u)$ satisfying the Δ_2-condition such that the inclusion $L_N \subset L_p$ is strong and $H \subset L_N$;*
(iii) *there is an Orlicz function $N(u)$ such that the inclusion $L_N \subset L_p$ is strong and $H \subset L_N$.*

Remark 1. It is worth to note that in the case of general $\Lambda(p)$-spaces we cannot prove a result similar to Bachelis–Ebenstein theorem mentioned in the Introduction (see [3]). Indeed, let $1 \leqslant p < \infty$ and let f_0 be a mean zero function independent of the sequence of Rademacher functions $\{r_k\}_{k=1}^\infty$ such that $f_0 \in L_p \setminus \bigcup_{q>p} L_q$. Then, the $\Lambda(p)$-space H spanned by the set $\{f_0\} \cup \{r_k\}_{k=1}^\infty$ is not contained in $L_{p'}$ and therefore it is not a $\Lambda(p')$-space for any $p' > p$. Moreover, it is not hard to check that the unit ball of H has equi-absolutely continuous norms in L_p and hence, by Proposition 1, there is an Orlicz function $N(u)$ such that the inclusion $L_N \subset L_p$ is strong and $H \subset L_N$.

3. $\Lambda(p)$-spaces and inclusions of r.i. spaces

Let $1 \leqslant p < \infty$, and let an r.i. space X on $[0,1]$ be embedded into $L_p = L_p[0,1]$. Suppose that the (canonic) inclusion operator $I : X \to L_p$ is strong but not strictly singular, and therefore there exists a subspace H of L_p such that the norms of X and L_p are equivalent on H. Then, it is easy to check that the unit ball $B_H = \{f \in H\colon \|f\|_p \leqslant 1\}$ has equi-absolutely continuous norms in L_p. As we will see further, it is not the case, in general, when the inclusion operator is only DSS. However, the following result holds.

Theorem 1. *Let $1 \leqslant p < \infty$, and let H be a closed subspace of L_p. Suppose that there is an r.i. space X such that $H \subset X \subset L_p$ and the inclusion operator $I : X \to L_p$ is DSS. Then H is a $\Lambda(p)$-space.*

Before proving this theorem let us recall the following notation, which has its origin in the classical Kadec–Pełczyński work [15] (see also [24]). For a given r.i. space X on $[0,1]$ and any $\delta > 0$ we set

$$S_{X,\delta}(f) := \bigl\{t \in [0,1]\colon \bigl|f(t)\bigr| > \delta\|f\|_X\bigr\}, \quad \text{where } f \in X,$$

and

$$M_{X,\delta} := \bigl\{f \in X\colon m\bigl(S_{X,\delta}(f)\bigr) \geqslant \delta\bigr\}.$$

It is not hard to check that for arbitrary set $K \subset X$ the following conditions are equivalent:

(a) there is a constant $C > 0$ such that for every $f \in K$ we have $\|f\|_X \leqslant C\|f\|_{1/2}$, where $\|f\|_{1/2} = (\int_0^1 |f(t)|^{1/2}\,dt)^2$;
(b) there exists $\delta > 0$ such that $K \subset M_{X,\delta}$.

The following result is actually well known (see e.g. [22, Proposition 1.c.8 and its proof] or [1, Lemma 5.2.1]) and therefore its proof is omitted.

Lemma 5. *Let X be a separable r.i. space and let K be a subset of the unit ball of X. If $K \not\subset M_{X,\delta}$ for every $\delta > 0$, then there exist $f_n \in K$, $n = 1, 2, \ldots$, and pairwise disjoint sets $A_n \subset [0,1]$, $n = 1, 2, \ldots$, such that $\lim_{n\to\infty} \|f_n - f_n\chi_{A_n}\|_X = 0$.*

Proof of Theorem 1. Since H is closed in L_p, the inclusion $X \subset L_p$ is continuous, and $H \subset X$, the norms of X and L_p are equivalent on H. Hence, without loss of generality, it can be assumed that, for some constant $c_0 \in (0,1)$, we have

$$\|f\|_p \leqslant \|f\|_X \quad (f \in X) \quad \text{and} \quad c_0\|f\|_X \leqslant \|f\|_p \quad (f \in H). \tag{5}$$

In particular, from inequalities (5) it follows that $H \subset X_\circ$, where $X_\circ$ is the separable part of X (see Section 2). Therefore, it is sufficient to prove the assertion in the case of separable X.

Assuming the contrary, by the definition of a $\Lambda(p)$-space, we deduce that the above condition (a) does not hold for $X = L_p$ and $K = B_H$, where $B_H := \{f \in H: \|f\|_p \leqslant 1\}$. Therefore, from the remark preceding Lemma 5 it follows that $B_H \not\subset M_{L_p,\delta}$ for arbitrary $\delta > 0$. Hence, taking into account (5), it is easy to see that the set $\{f \in H: \|f\|_X = 1\}$ is not contained in $M_{X,\delta}$ for every $\delta > 0$. Applying Lemma 5 to this set, we can find a sequence of functions $\{f_k\}_{k=1}^\infty \subset H$, $\|f_k\|_X = 1$, and pairwise disjoint subsets $A_k \subset [0,1]$, $k = 1,2,\ldots$, satisfying the inequalities

$$\|f_k - f_k\chi_{A_k}\|_X \leqslant c_0 2^{-k-4}, \quad k = 1,2,\ldots$$

Combining this with (5), for all $k = 1,2,\ldots$ we infer

$$\|f_k\chi_{A_k}\|_p \geqslant \|f_k\|_p - \|f_k - f_k\chi_{A_k}\|_p \geqslant c_0 - \|f_k - f_k\chi_{A_k}\|_X \geqslant c_0\bigl(1 - 2^{-k-4}\bigr),$$

and similarly

$$\|f_k\chi_{A_k}\|_X \geqslant 1 - c_0 2^{-k-4}.$$

Therefore, setting $g_k := \frac{f_k\chi_{A_k}}{\|f_k\chi_{A_k}\|_X}$ $(k = 1,2,\ldots)$, we have

$$\|f_k - g_k\|_X \leqslant \|f_k - f_k\chi_{A_k}\|_X + \|f_k\chi_{A_k} - g_k\|_X \leqslant c_0 2^{-k-4} + 1 - \|f_k\chi_{A_k}\|_X \leqslant c_0 2^{-k-3}$$

and

$$\|g_k\|_p = \frac{\|f_k\chi_{A_k}\|_p}{\|f_k\chi_{A_k}\|_X} \geqslant c_0\bigl(1 - 2^{-k-4}\bigr) \geqslant \frac{c_0}{2}, \quad k = 1,2,\ldots$$

Hence, by the first inequality in (5), we obtain

$$\sum_{k=1}^\infty \frac{\|f_k - g_k\|_p}{\|g_k\|_p} \leqslant \sum_{k=1}^\infty \frac{\|f_k - g_k\|_X}{\|g_k\|_p} \leqslant \frac{1}{4}$$

and

$$\sum_{k=1}^\infty \|f_k - g_k\|_X \leqslant \frac{1}{4}.$$

Since $\{g_k\}$ is a basic sequence both in X and L_p, from the last inequalities and the well-known stability property (see e.g. [1, Theorem 1.3.9]) it follows that $\{f_k\}$ is also a basic sequence equivalent to the sequence $\{g_k\}$ in both spaces X and L_p. Therefore, by the latter inequality in (5), we have

$$\left\|\sum_{k=1}^\infty a_k g_k\right\|_X \leqslant C\left\|\sum_{k=1}^\infty a_k f_k\right\|_X \leqslant C\left\|\sum_{k=1}^\infty a_k f_k\right\|_p \leqslant C\left\|\sum_{k=1}^\infty a_k g_k\right\|_p,$$

for any $a_k \in \mathbb{R}$, whence the inclusion operator $I : X \to L_p$ is an isomorphism on the closed linear span $[g_k]$ in X. Since the functions g_k are pairwise disjoint, we conclude that I is not DSS. This contradiction finishes the proof. □

In the next section, we prove that, if $1 \leqslant p < 2$, the unit ball B_H of every $\Lambda(p)$-space H has equi-absolutely continuous norms in L_p and therefore, by Proposition 1, there is an Orlicz function N such that the inclusion $L_N \subset L_p$ is strong and $H \subset L_N$. The situation is completely different in the case when $p \geqslant 2$. Applying the above-mentioned Bourgain's result [7], in Example 1, we show that for every $p > 2$ there is a $\Lambda(p)$-space, which is not embedded into any Orlicz space L_N such that $L_N \overset{\neq}{\subset} L_p$. Furthermore, in Example 2, we construct a $\Lambda(2)$-space spanned by a sequence of mean zero independent functions, which is not embedded into any r.i. space X such that $X \overset{\neq}{\subset} L_2$.

Example 1. Let $p > 2$, and let $|A|$ be the number of elements of a finite set A, $[z]$ be the integer part of $z \in \mathbb{R}$. By Theorem 1 in [7], for any $k = 1, 2, \ldots$ there is a set $S_k \subset \{n \in \mathbb{Z}\colon\ 2^k \leqslant n < 2^{k+1}\}$ such that

$$|S_k| = \left[4^{k/p}\right] \tag{6}$$

and

$$\left\|\sum_{j \in S_k} a_j e^{2\pi i j x}\right\|_p \leqslant C\left(\sum_{j \in S_k} |a_j|^2\right)^{1/2} \quad (k = 1, 2, \ldots) \tag{7}$$

for some $C > 0$, which depends only on p, and for all $a_j \in \mathbb{R}$.

As in the proof of Theorem 2 in [7], we set $S := \bigcup_{k=1}^{\infty} S_k$. Let us consider the subspace $H = [e^{2\pi i j x}]_{j \in S}$ in $L_p = L_p[-\pi, \pi]$. By the Littlewood–Paley inequality (see e.g. [33, Chapter XV, Theorem 4.22]), we have

$$\left\|\sum_{j \in S} a_j e^{2\pi i j x}\right\|_p \asymp \left\|\left(\sum_{k=1}^{\infty}\left(\sum_{j \in S_k} a_j e^{2\pi i j x}\right)^2\right)^{1/2}\right\|_p.$$

Hence, from (7) and Minkowski's inequality it follows that

$$\left\|\sum_{j \in S} a_j e^{2\pi i j x}\right\|_p \leqslant C\left(\sum_{k=1}^{\infty}\left\|\sum_{j \in S_k} a_j e^{2\pi i j x}\right\|_p^2\right)^{1/2} \asymp \left(\sum_{j=1}^{\infty} |a_j|^2\right)^{1/2}.$$

Since

$$\left\|\sum_{j \in S} a_j e^{2\pi i j x}\right\|_2 = \sqrt{2\pi}\left(\sum_{j=1}^{\infty} |a_j|^2\right)^{1/2},$$

we conclude that H is a $\Lambda(p)$-space.

Let us show that the space H possesses the following property: if X is an r.i. space such that $X \subset L_p$ and $H \subset X$, then its fundamental function $\varphi_X(t)$ is equivalent to the function $t^{1/p}$ on $[0,1]$. Indeed, denote $f_k := 2^{-k/p} \sum_{j \in S_k} e^{2\pi ijx}$, $k = 1, 2, \ldots$. Then, by hypothesis and inequality (7), we have $\|f_k\|_X \leqslant C\|f_k\|_p \leqslant C$ for all $k = 1, 2, \ldots$. Therefore, in view of (6) and the quasi-concavity of φ_X, we obtain

$$\begin{aligned} C &\geqslant \|f_k \chi_{[-2^{-k}/10, 2^{-k}/10]}\|_X \geqslant c2^{-k/p}|S_k| \|\chi_{[-2^{-k}/10, 2^{-k}/10]}\|_X \\ &\geqslant c2^{k/p} \varphi_X\left(2^{-k}\right) \quad (k = 1, 2, \ldots), \end{aligned}$$

whence $\varphi_X(t) \leqslant Ct^{1/p}$. Since the opposite inequality follows from the embedding $X \subset L_p$, the proof is complete.

The last property implies that H cannot be embedded into any Orlicz space L_N such that $L_N \overset{\neq}{\subset} L_p$. Indeed, if $L_N \subset L_p$ and $H \subset L_N$, then, as is proved, $\varphi_{L_N}(t) \asymp t^{1/p}$ $(0 \leqslant t \leqslant 1)$. Hence, since $\varphi_{L_N}(t) = 1/N^{-1}(1/t)$, it follows that for some $C_1 > 0$, $C_2 > 0$, and all large $t > 0$ we have

$$N(C_1 t) \leqslant t^{1/p} \leqslant N(C_2 t).$$

Therefore, $L_N = L_p$ (with equivalence of norms), and our assertion is proved.

Example 2. Let $\{f_k\}_{k=1}^\infty$ be a sequence of independent functions on $[0,1]$ such that $\int_0^1 f_k(t)\,dt = 0$, $|f_k(t)| = 2^{k/2}$, $t \in E_k$, where $m(E_k) = 2^{-k-1}$, and $|f_k(t)| = 1$, $t \in [0,1] \setminus E_k$ $(k = 1, 2, \ldots)$. Let us prove that $H := [f_k]$ is a $\Lambda(2)$-space. Since f_k are mean zero independent functions, and $\|f_k\|_2 \asymp 1$, $k = 1, 2, \ldots$, we have

$$\left\| \sum_{k=1}^\infty c_k f_k \right\|_2 \asymp \left\|(c_k)\right\|_2. \tag{8}$$

Thus, it suffices to check only that the following inequality holds:

$$\left\| \sum_{k=1}^\infty c_k f_k \right\|_1 \geqslant c\left\|(c_k)\right\|_2. \tag{9}$$

By using the independence of f_k, from [14, Theorem 1] it follows

$$\left\| \sum_{k=1}^\infty c_k f_k \right\|_1 \asymp \left\| \sum_{k=1}^\infty c_k \bar{f}_k \chi_{A_k} \right\|_1 + \left\| \sum_{k=1}^\infty c_k \bar{f}_k \chi_{B_k} \right\|_2 \quad \text{for all } c_k \in \mathbb{R},$$

where $\bar{f}_k$ are disjointly supported copies (defined on $[0,\infty)$) of the functions f_k, $\{A_k\}_{k=1}^\infty$ and $\{B_k\}_{k=1}^\infty$ are sequences of pairwise disjoint sets such that $\bigcup_{k=1}^\infty A_k = [0,1]$, $\bigcup_{k=1}^\infty B_k = [1,\infty)$, $m(A_k \cup B_k) = 1$ $(k = 1, 2, \ldots)$, and the function $\sum_{k=1}^\infty c_k \bar{f}_k \chi_{A_k}$ (resp. $\sum_{k=1}^\infty c_k \bar{f}_k \chi_{B_k}$) is equimeasurable with the function $(\sum_{k=1}^\infty c_k \bar{f}_k)^* \chi_{[0,1]}$ (resp.

$(\sum_{k=1}^{\infty} c_k \bar{f}_k)^* \chi_{[1,\infty)})$ $(k = 1, 2, \ldots)$. Further, we consider two cases separately: (a) there is a k_0 such that $m(A_{k_0}) > 1/2$; (b) $m(A_k) \leqslant 1/2$ for all $k = 1, 2, \ldots$. Clearly, in the case (a) we have $m(B_k) \geqslant 1/2$ for all $k \neq k_0$. Hence, by definition of f_k's, we obtain

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} c_k f_k\right\|_1 &\geqslant c\left(|c_{k_0}| \|\bar{f}_{k_0} \chi_{A_{k_0}}\|_1 + \left\|\sum_{k \neq k_0} c_k \bar{f}_k \chi_{B_k}\right\|_2\right) \\ &\geqslant c\left(\frac{1}{2}|c_{k_0}| + \left(\sum_{k \neq k_0} c_k^2 \|\bar{f}_k \chi_{B_k}\|_2^2\right)^{1/2}\right) \\ &\geqslant c\left(|c_{k_0}| + \left(\sum_{k \neq k_0} c_k^2\right)^{1/2}\right) \geqslant c\|(c_k)\|_2.\end{aligned}$$

In the case (b) we have $m(B_k) \geqslant 1/2$, $k = 1, 2, \ldots$, whence

$$\left\|\sum_{k=1}^{\infty} c_k f_k\right\|_1 \geqslant c\left\|\sum_{k=1}^{\infty} c_k \bar{f}_k \chi_{B_k}\right\|_2 = c\left(\sum_{k=1}^{\infty} c_k^2 \|\bar{f}_k \chi_{B_k}\|_2^2\right)^{1/2} \geqslant c\|(c_k)\|_2.$$

Thus, inequality (9) is proved, and hence H is a $\Lambda(2)$-space. Show that H cannot be embedded into any r.i. space X such that $X \overset{\neq}{\subset} L_2$.

Suppose X is an r.i. space such that $X \subset L_2$ and $H \subset X$. Applying [14, Theorem 1] once more (this time to the space X), for all $c_k \in \mathbb{R}$ we have

$$\left\|\sum_{k=1}^{\infty} c_k f_k\right\|_X \geqslant c\left(\left\|\sum_{k=1}^{\infty} c_k \bar{f}_k \chi_{A_k}\right\|_X + \left\|\sum_{k=1}^{\infty} c_k \bar{f}_k \chi_{B_k}\right\|_2\right),$$

where the functions $\bar{f}_k$ and the sets $\{A_k\}_{k=1}^{\infty}$, $\{B_k\}_{k=1}^{\infty}$ are defined as above. Let $c_k > 0$ be such that the sequence $(c_k 2^{k/2})_{k=1}^{\infty}$ increases. Since $\sum_{k=1}^{\infty} c_k \bar{f}_k \chi_{A_k}$ is equimeasurable with the function $(\sum_{k=1}^{\infty} c_k \bar{f}_k)^* \chi_{[0,1]}$, then taking into account definition of f_k's, we obtain

$$\left\|\sum_{k=1}^{\infty} c_k f_k\right\|_X \geqslant c\left\|\sum_{k=1}^{\infty} c_k 2^{k/2} \chi_{(2^{-k-1}, 2^{-k}]}\right\|_X.$$

By hypothesis, the norms of X and L_2 are equivalent on H. Therefore, from the preceding inequality, the embedding $X \subset L_2$, and the equivalence

$$\left\|\sum_{k=1}^{\infty} c_k f_k\right\|_2 \asymp \left\|\sum_{k=1}^{\infty} c_k 2^{k/2} \chi_{(2^{-k-1}, 2^{-k}]}\right\|_2$$

it follows that

$$\left\|\sum_{k=1}^{\infty} a_k \chi_{(2^{-k-1},2^{-k}]}\right\|_X \asymp \left\|\sum_{k=1}^{\infty} a_k \chi_{(2^{-k-1},2^{-k}]}\right\|_2 \tag{10}$$

for every positive increasing sequence $(a_k)_{k=1}^{\infty}$ with constants, which depend only on X. On the other hand, for arbitrary r.i. space Y on $[0,1]$ the following equivalence holds

$$\|y\|_Y \asymp \left\|\sum_{k=1}^{\infty} y^*\left(2^{-k}\right)\chi_{(2^{-k-1},2^{-k}]}\right\|_Y,$$

with constants independent of $y \in Y$. Hence, from (10) it follows that $X = L_2$ (with equivalence of norms).

4. Unit ball of a $\Lambda(p)$-space, $1 \leqslant p < \infty$

By Lemma 2, for any $1 \leqslant p < \infty$ every closed subspace of L_p whose unit ball has equi-absolutely continuous norms in L_p is a $\Lambda(p)$-space. Here, we will consider the question when the converse implication holds. From Proposition 1 and Examples 1 and 2 it follows that the answer is negative if $p \geqslant 2$. At the same time, for the other values of p we have the following positive result (a close statement follows also from the results, which were announced in the short paper [24] and proved then in [25]).

Theorem 2. *If H is a $\Lambda(p)$-space, where $1 \leqslant p < 2$, then its unit ball B_H has equi-absolutely continuous norms in L_p.*

Proof. By [3, Lemma 2], an arbitrary $\Lambda(1)$-set $E \subset \mathbb{Z}$ generates the reflexive space L_E. Using the same proof, this result can be extended to all $\Lambda(1)$-spaces. Therefore, if $p = 1$, from the Dunford–Pettis theorem (see e.g. [1, Theorem 5.2.9]) it follows that the unit ball B_H has equi-absolutely continuous norms in L_1. Thus, further we can confine ourselves to the case $1 < p < 2$.

Let us assume that B_H fails to have equi-absolutely continuous norms in L_p. Then

$$\alpha := \frac{1}{4}\lim_{\delta\to 0}\sup_{m(E)<\delta}\sup_{f\in B_H}\|f\chi_E\|_p > 0.$$

We construct a bounded unconditional basic sequence $\{f_i\}_{i=1}^{\infty}$ in L_p, $f_i \in H$ $(i = 1, 2, \ldots)$, such that for some pairwise disjoint sets $F_i \subset [0,1]$ it holds:

$$\|f_i\chi_{F_i}\|_p \geqslant \alpha \quad (i = 1, 2, \ldots). \tag{11}$$

First, we define inductively auxiliary sequences of functions $\{g_n\}_{n=1}^{\infty} \subset H$ and sets $\{E'_n\}_{n=1}^{\infty}$ as follows. Let $\eta_1 = 1$, and let $E'_1 \subset [0,1]$ and $g_1 \in H$ be such that $m(E'_1) < \eta_1/2$, $\|g_1\|_p = 1$, and $\|g_1\chi_{E'_1}\|_p \geqslant 3\alpha$. If $\eta_2 \in (0, \eta_1/2)$ is sufficiently small, then $\|g_1\chi_A\|_p \leqslant \alpha$ whenever $m(A) \leqslant \eta_2$. Moreover, there are a set $E_2' \subset [0,1]$, with $m(E_2') < \eta_2/2$, and a function $g_2 \in H$, with $\|g_2\|_p = 1$, for which we have $\|g_2\chi_{E_2'}\|_p \geqslant 3\alpha$.

Assume that the functions $g_1, \ldots, g_{n-1}$ from H, with $\|g_i\|_p = 1$, the sets $E_1', \ldots, E_{n-1}'$, and the numbers $\eta_1, \ldots, \eta_{n-1}$ such that $\eta_i \in (0, \eta_{i-1}/2)$ $(i = 2, \ldots, n-1)$, $m(E_i') < \eta_i/2$ $(i = 1, \ldots, n-1)$, $\|g_i\chi_{E_i'}\|_p \geqslant 3\alpha$ and $\|g_j\chi_A\|_p \leqslant \alpha$ for all $j = 1, \ldots, i-1$ whenever $m(A) \leqslant \eta_i$ $(i = 1, \ldots, n-1)$ are already chosen. Then there exists $\eta_n \in (0, \eta_{n-1}/2)$ with the property that $\|g_j\chi_A\|_p \leqslant \alpha$ for all $1 \leqslant j \leqslant n-1$ and for any set A such that $m(A) \leqslant \eta_n$. Using hypothesis once again, we find a function $g_n \in H$, $\|g_n\|_p = 1$, and a set $E_n' \subset [0,1]$, $m(E_n') < \eta_n/2$, which satisfy the condition $\|g_n\chi_{E_n'}\|_p \geqslant 3\alpha$.

Now, let us define the pairwise disjoint sets $E_n := E_n' \setminus \bigcup_{i=n+1}^{\infty} E_i'$, $n = 1, 2, \ldots$. Since

$$m\left(\bigcup_{i=n+1}^{\infty} E_i'\right) \leqslant \sum_{i=n+1}^{\infty} m(E_i') \leqslant \frac{1}{2}\sum_{i=n+1}^{\infty} \eta_i \leqslant \frac{1}{2}\sum_{i=1}^{\infty} \frac{\eta_{n+1}}{2^{i-1}} = \eta_{n+1},$$

in view of the choice of η_{n+1} we have

$$\|g_n\chi_{E_n}\|_p \geqslant \|g_n\chi_{E_n'}\|_p - \|g_n\chi_{\bigcup_{i=n+1}^{\infty} E_i'}\|_p \geqslant 2\alpha, \quad n = 1, 2, \ldots$$

Hence,

$$\|g_n\chi_{E_n}\|_p \geqslant 2\alpha \quad \text{and} \quad \|g_j\chi_{E_n}\|_p \leqslant \alpha, \quad 1 \leqslant j \leqslant n-1, \ n = 1, 2, \ldots \tag{12}$$

Let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary unconditional basis in L_p (say, the Haar system), and let $\{x_n^*\}_{n=1}^{\infty}$ be the sequence of biorthogonal functionals associated with $\{x_n\}_{n=1}^{\infty}$. Since $\|g_n\|_p = 1$ $(n = 1, 2, \ldots)$, for every $k = 1, 2, \ldots$ we can find a subsequence $\{g_n^{(k)}\} \subset \{g_n\}$ such that there is the limit $\lim_{n\to\infty} x_k^*(g_n^{(k)})$. Applying the diagonal method, select a further subsequence $\{g_{n_i}\} \subset \{g_n\}$, for which $\lim_{i\to\infty} x_k^*(g_{n_i})$ exists for arbitrary $k = 1, 2, \ldots$. Then, setting $f_i := g_{n_{i+1}} - g_{n_i}$, we obtain that $\lim_{i\to\infty} x_k^*(f_i) = 0$ $(k = 1, 2, \ldots)$. Therefore, by [21, Proposition 1.a.12], we can find a subsequence of the sequence $\{f_i\}$ (we still denote it by $\{f_i\}$), which is equivalent to some block basis of $\{x_n\}_{n=1}^{\infty}$ in L_p, and hence it is also an unconditional basic sequence in this space. Moreover, $\|f_i\|_p \leqslant 2$ for all $i = 1, 2, \ldots$, and from (12) it follows that

$$\|f_i\chi_{E_{n_{i+1}}}\|_p \geqslant \|g_{n_{i+1}}\chi_{E_{n_{i+1}}}\|_p - \|g_{n_i}\chi_{E_{n_{i+1}}}\|_p \geqslant \alpha.$$

As a result, inequalities (11) hold if $F_i := E_{n_{i+1}}$ $(i = 1, 2, \ldots)$.

Thanks to unconditionality of $\{f_i\}$ and inequalities (11), we can apply Lemma 2 from [13] and deduce that the sequence $\{f_i\}$ is equivalent to the standard basis of l_p. Therefore, since $1 < p < 2$, by Rosenthal's theorem [27, Theorem 13], we conclude that H fails to be a $\Lambda(p)$-space. This contradiction finishes the proof. $\square$

Next, we consider the case $p = 2$. It turns out that, if a $\Lambda(2)$-space is obtained by a special way, its unit ball has equi-absolutely continuous norms in L_2. More precisely, the following result holds.

Theorem 3. *Let X be an r.i. space on $[0,1]$ such that $X \subset L_2$, and let the norms of L_2 and X be equivalent on some closed subspace H of L_2. Then, if the inclusion operator $I: X \to L_2$ is DSS, the unit ball B_H of H has equi-absolutely continuous norms in L_2.*

Proof. Assuming that B_H fails to have equi-absolutely continuous norms in L_2 and arguing in the same way as in the proof of Theorem 2, we find a bounded unconditional basic sequence $\{f_i\}_{i=1}^{\infty}$ in L_2, $f_i \in H$ $(i = 1, 2, \ldots)$, such that

$$\|f_i \chi_{F_i}\|_2 \geqslant \alpha \quad (i = 1, 2, \ldots) \tag{13}$$

for some $\alpha > 0$ and pairwise disjoint sets $F_i \subset [0,1]$. Show that the norms of X and L_2 are equivalent on the subspace $[f_n \chi_{F_n}]_{n=1}^{\infty}$.

At first, as above, taking into account unconditionality of $\{f_i\}$ and (13), we can apply Lemma 2 from [13] and deduce that the sequence $\{f_i\}$ is equivalent to the standard basis of l_2. Therefore, since $\{f_n\} \subset H$, by condition, we have

$$\left\| \sum_{n=1}^{\infty} c_n f_n \right\|_X \asymp \left\| \sum_{n=1}^{\infty} c_n f_n \right\|_2 \asymp \|(c_n)\|_2.$$

Hence, since the sets F_n are pairwise disjoint, by Khintchine inequality for L_1 with the optimal constant [29], we infer

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} c_n f_n \chi_{F_n} \right\|_X &\leqslant \left\| \left(\sum_{n=1}^{\infty} c_n^2 f_n^2 \right)^{1/2} \right\|_X \\ &\leqslant \sqrt{2} \left\| \int_0^1 \left| \sum_{n=1}^{\infty} c_n f_n r_n(s) \right| ds \right\|_X \\ &\leqslant \sqrt{2} \int_0^1 \left\| \sum_{n=1}^{\infty} c_n f_n r_n(s) \right\|_X ds \leqslant C \|(c_n)\|_2. \end{aligned}$$

Moreover, from (13) it follows that

$$\left\| \sum_{n=1}^{\infty} c_n f_n \chi_{F_n} \right\|_2 = \left(\sum_{n=1}^{\infty} |c_n|^2 \|f_n \chi_{F_n}\|_2^2 \right)^{1/2} \geqslant \alpha \|(c_n)\|_2.$$

Therefore, we have

$$\left\| \sum_{n=1}^{\infty} c_n f_n \chi_{F_n} \right\|_X \leqslant C \left\| \sum_{n=1}^{\infty} c_n f_n \chi_{F_n} \right\|_2 .$$

On the other hand, from the embedding $X \subset L_2$ it follows

$$\left\| \sum_{n=1}^{\infty} c_n f_n \chi_{F_n} \right\|_2 \leqslant C \left\| \sum_{n=1}^{\infty} c_n f_n \chi_{F_n} \right\|_X ,$$

and equivalence of the norms of X and L_2 on the subspace $[f_n \chi_{F_n}]_{n=1}^{\infty}$ is proved. Hence, the inclusion operator $I : X \to L_2$ fails to have DSS-property. This contradicts the hypothesis, and theorem is proved. □

Remark 2. Let us show that a modification of the proof of Theorem 2, allows to construct, in the case $p = 2$, even an orthonormal sequence $\{f_i\}_{i=1}^{\infty} \subset H$ satisfying inequalities (13) for some pairwise disjoint sets $F_i \subset [0, 1]$, $i = 1, 2, \ldots$.

Denoting by B_n $(n = 1, 2, \ldots)$ the maximum of all determinants of order n whose elements have absolute value not exceeding 1 and arguing in the same way as above, we can find sequences of functions $\{g_n\}_{n=1}^{\infty} \subset H$ and pairwise disjoint sets $\{F_n\}_{n=1}^{\infty}$ such that $\|g_n\|_2 = 1$,

$$\|g_n \chi_{F_n}\|_2 \geqslant 2\alpha, \quad \text{and} \quad \|g_j \chi_{F_n}\|_2 \leqslant n^{-1} B_n^{-1} \alpha, \quad 1 \leqslant j \leqslant n-1, \ n = 1, 2, \ldots \tag{14}$$

Now, the sequence $\{f_n\}$ can be obtained by applying the standard orthogonalization method to the sequence $\{g_n\}$. Namely, denoting by (f, g) the inner product in $L_2 = L_2[0, 1]$, we set

$$f'_n = \sum_{k=1}^{n} \alpha_k^{(n)} g_k = \begin{vmatrix} (g_1, g_1) & (g_1, g_2) & \cdots & (g_1, g_{n-1}) & g_1 \\ (g_2, g_1) & (g_2, g_2) & \cdots & (g_2, g_{n-1}) & g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (g_n, g_1) & (g_n, g_2) & \cdots & (g_n, g_{n-1}) & g_n \end{vmatrix}$$

and $f_n := f'_n / \|f'_n\|_2$ $(n = 1, 2, \ldots)$ (coefficients $\alpha_k^{(n)}$ are calculated by formal expansion of the determinant with respect to the last column). Since $\|g_n\|_2 = 1$ $(n = 1, 2, \ldots)$, from the Cauchy–Schwarz–Bunyakovskii inequality it follows that $|\alpha_k^{(n)}| \leqslant B_{n-1}$. On the other hand, as is known,

$$f_1 = g_1, \qquad f'_n = g_n - \sum_{k=1}^{n-1} (g_n, f_k) f_k, \quad n \geqslant 2,$$

whence $\alpha_n^{(n)} = 1$ $(n = 1, 2, \ldots)$. In addition, since the sum $\sum_{k=1}^{n-1} (g_n, f_k) f_k$ is the orthogonal projection of g_n to the subspace $[f_k]_{k=1}^{n-1}$, we have $\|f'_n\|_2 \leqslant \|g_n\|_2 \leqslant 1$. Therefore, by (14),

$$\|f_n\chi_{F_n}\|_2 \geqslant \|f'_n\|_2^{-1}\left(\|g_n\chi_{F_n}\|_2 - \sum_{k=1}^{n-1}|\alpha_k^{(n)}|\|g_k\chi_{F_n}\|_2\right) \geqslant \alpha.$$

Thus, $f_n \in H$ and $F_n \subset [0,1]$ $(n = 1,2,\dots)$ satisfy inequalities (13).

The following result is an immediate consequence of Theorem 3 and Proposition 1.

Corollary 1. *Let H be a closed subspace of L_2 such that there is an r.i. space $X \subset L_2$ with $H \subset X$. If the inclusion operator $I : X \to L_2$ is DSS, then there is a 2-convex Orlicz function N satisfying the Δ_2-condition such that the embedding $L_N \subset L_2$ is strong and $H \subset L_N$.*

Now, we show that in the case $p > 2$ the unit ball of a $\Lambda(p)$-space, in general, fails to have equi-absolutely continuous norms in L_p even if there is an r.i. space X such that $H \subset X \subset L_p$ and the inclusion operator $I : X \to L_p$ is DSS.

Theorem 4. *For arbitrary $p > 2$ there exists an Orlicz space L_F with the following properties: $L_F \subset L_p$, the inclusion operator $I : L_F \to L_p$ is DSS, and the norms of L_p and L_F are equivalent on some $\Lambda(p)$-space H whose unit ball B_H fails to have equi-absolutely continuous norms in L_p.*

Proof. Let $q \in \mathbb{R}$ be arbitrary. Following [11] (see also [12, p. 236]), we consider the function $F_{p,q}$, which is defined as follows: $F_{p,q}(0) = 0$ and $F_{p,q}(t) = t^p e^{qf(\ln t)}$ if $t > 0$, where

$$f(x) := \sum_{k=1}^{\infty}\bigl(1 - \cos\bigl(\pi x 2^{-k}\bigr)\bigr). \tag{15}$$

It is not hard to check (see also [11, Lemma 1.1]) that for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for any $s > 0$

$$C_\varepsilon^{-1} t^{p-|q|\varepsilon} \leqslant \frac{F_{p,q}(st)}{F_{p,q}(s)} \leqslant C_\varepsilon t^{p+|q|\varepsilon} \quad \text{if } t \geqslant 1, \tag{16}$$

and

$$C_\varepsilon^{-1} t^{p+|q|\varepsilon} \leqslant \frac{F_{p,q}(st)}{F_{p,q}(s)} \leqslant C_\varepsilon t^{p-|q|\varepsilon} \quad \text{if } 0 < t < 1. \tag{17}$$

In particular, these inequalities imply that $F_{p,q}$ is equivalent to a convex function on $[0,\infty)$. Moreover, since $f(x) \geqslant 0$ for all $x \in \mathbb{R}$, we have $t^p \leqslant F_{p,q}(t)$ $(t \geqslant 0)$. Hence, the Orlicz space $L_{F_{p,q}}$ is embedded into L_p and

$$\|f\|_p \leqslant \|f\|_{L_{F_{p,q}}} \quad (f \in L_{F_{p,q}}). \tag{18}$$

At the same time, for every $n \in \mathbb{N}$ we have

$$f(2^n) = \sum_{k=1}^{\infty}\bigl(1 - \cos\bigl(\pi 2^{n-k}\bigr)\bigr) = \sum_{k=1}^{\infty}\bigl(1 - \cos\bigl(\pi 2^{-k+1}\bigr)\bigr)$$
$$= 2\sum_{k=1}^{\infty}\sin^2\bigl(\pi 2^{-k}\bigr) \leqslant 2\pi^2\sum_{k=1}^{\infty}2^{-2k} = \frac{2\pi^2}{3}.$$

Therefore, if $a_n := e^{2^n}$, then

$$F_{p,q}(a_n) \leqslant C_q a_n^p \quad (n \in \mathbb{N}), \tag{19}$$

with some constant $C_q > 0$ depending only on q. Setting $b_n := 1/F_{p,q}(a_n)$ $(n \in \mathbb{N})$, we see that $b_n \to 0$ as $n \to \infty$. Also, since the fundamental function of an Orlicz space L_F is calculated by the formula $\varphi_{L_F}(u) = 1/F^{-1}(1/u)$ (F^{-1} is the inverse function to F) [19, Chapter 2, §2], we have $\varphi_{L_{F_{p,q}}}(b_n) = 1/a_n$ $(n \in \mathbb{N})$. Hence, by inequality (19), we obtain

$$\varphi_{L_{F_{p,q}}}(b_n) \leqslant C_q^{1/p}\varphi_{L_p}(b_n), \quad n \in \mathbb{N}. \tag{20}$$

Thus, the inclusion $L_{F_{p,q}} \subset L_p$ is not strong. On the other hand, in [11], it is proved that under a certain condition on q the inclusion operator $I : L_{F_{p,q}} \to L_p$ is DSS. Namely, we set

$$\gamma_n^f := \inf_{s>0}\max_{0\leqslant x,y\leqslant 2^n}\bigl(f(x+s) - f(y+s)\bigr), \quad n = 1, 2, \ldots,$$

where the function f is still defined by (15). Then $\gamma_f := \liminf_{n\to\infty}\gamma_n^f n^{-1} > 0$, and if $q > 2p\ln 2(\gamma_f)^{-1}$ the Orlicz space $L_{F_{p,q}}$ does not contain any complemented subspace spanned by disjointly supported functions and isomorphic to l_p [11, Lemma 1.5 and Corollary 1.8]. Hence, taking into account Proposition 3.3 and Corollary 3.4 from the same paper, we deduce that I is disjointly strictly singular.

Finally, we show that the subspace H spanned by a sequence of independent functions $\{f_k\}_{k=1}^{\infty}$ on $[0, 1]$ such that $\int_0^1 f_k(t)\,dt = 0$, $|f_k(t)| = b_k^{-1/p}$ if $t \in E_k$, where $m(E_k) = b_k$, and $|f_k(t)| = 1$ if $t \in [0, 1] \setminus E_k$ $(k = 1, 2, \ldots)$, satisfies all required conditions. First, let us prove that such a sequence is equivalent to the standard basis of l_2 in both spaces L_p and $L_{F_{p,q}}$, i.e.,

$$\Bigl\|\sum_{k=1}^{\infty}c_k f_k\Bigr\|_p \asymp \Bigl\|\sum_{k=1}^{\infty}c_k f_k\Bigr\|_{L_{F_{p,q}}} \asymp \bigl\|(c_k)\bigr\|_2. \tag{21}$$

From (18) and (20) it follows that

$$\|f_k\|_p \asymp \|f_k\|_{L_{F_{p,q}}} \asymp 1, \quad k = 1, 2, \ldots \tag{22}$$

Furthermore, taking into account inequalities (16) and (17), we see that the dilation indices of the fundamental function $\varphi_{L_{F_{p,q}}}$ are equal to $1/p$. Therefore, $F_{p,q}$ satisfies the Δ_2-condition at infinity, and we can apply Theorem 1 from [14]. Then, for all $c_k \in \mathbb{R}$ we have

$$\left\|\sum_{k=1}^{\infty} c_k f_k\right\|_{L_{F_{p,q}}} \asymp \left\|\sum_{k=1}^{\infty} c_k \bar{f}_k \chi_{A_k}\right\|_{L_{F_{p,q}}} + \left\|\sum_{k=1}^{\infty} c_k \bar{f}_k \chi_{B_k}\right\|_2, \tag{23}$$

where $\bar{f}_k$ are disjointly supported copies of the functions f_k, $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ are sequences of pairwise disjoint sets such that $\bigcup_{k=1}^{\infty} A_k = [0,1]$, $\bigcup_{k=1}^{\infty} B_k = [1,\infty)$, $m(A_k \cup B_k) = 1$ $(k = 1,2,\ldots)$, and the function $\sum_{k=1}^{\infty} c_k \bar{f}_k \chi_{A_k}$ (resp. $\sum_{k=1}^{\infty} c_k \bar{f}_k \chi_{B_k}$) is equimeasurable with the function $(\sum_{k=1}^{\infty} c_k \bar{f}_k)^* \chi_{[0,1]}$ (resp. $(\sum_{k=1}^{\infty} c_k \bar{f}_k)^* \chi_{[1,\infty)}$) $(k = 1,2,\ldots)$. In view of the left hand side of (16) and the main result of the paper [16] the space $L_{F_{p,q}}$ satisfies an upper r-estimate for arbitrary $r < p$. Fixing any r such that $2 < r < p$ and using (23) and (22), we obtain

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} c_k f_k\right\|_{L_{F_{p,q}}} &\leqslant C_r'\left(\left(\sum_{k=1}^{\infty} |c_k|^r \|\bar{f}_k \chi_{A_k}\|_{L_{F_{p,q}}}^r\right)^{1/r} + \left(\sum_{k=1}^{\infty} c_k^2 \|\bar{f}_k \chi_{B_k}\|_2^2\right)^{1/2}\right) \\ &\leqslant C_r \big\|(c_k)\big\|_2. \end{aligned} \tag{24}$$

On the other hand, $b_k \leqslant a_k^{-p} \leqslant e^{-2}$ $(k = 1,2,\ldots)$, whence $f_k \in M_{L_p,\varepsilon}$ for all $k = 1,2,\ldots$ if $\varepsilon > 0$ is sufficiently small (see notation in the beginning of Section 4). Therefore, since $\{f_k\}$ is an unconditional sequence in L_p, by (22) and [15, Corollary 4], we conclude that $\{f_k\}$ is equivalent in L_p to the standard basis of l_2. Thus, equivalence (21) follows from inequalities (24) and (18).

Finally, from (21), DSS-property of the inclusion $L_{F_{p,q}} \subset L_p$, and Theorem 1 it follows that the subspace $H = [f_k]_{k=1}^{\infty}$ in L_p is a $\Lambda(p)$-space. At the same time, in view of (22) the relations

$$\|f_k \chi_{E_k}\|_p = 1 \quad (k = 1,2,\ldots) \quad \text{and} \quad m(E_k) = b_k \to 0 \quad \text{as } k \to \infty$$

show that the set B_H fails to have equi-absolutely continuous norms in L_p. $\square$

5. Uniformizable $\Lambda(2)$-spaces

In [6, Chapter III, Lemma 7], it is proved the following Hilbertian characterization of $\Lambda(2)$-spaces: a closed subspace H of L_2 is a $\Lambda(2)$-space if and only if there exists a constant $\delta > 0$ such that for every $f \in H$ there is a function $g \in H^{\perp}$, with $f + g \in L_\infty$ and $\|f + g\|_\infty \leqslant \delta \|f\|_2$ ($H^{\perp}$ is the orthogonal complement of H). In the same book it is introduced also a uniform variant of the last property (see Definition 6 in Chapter III). A closed subspace H of L_2 is called a *uniformizable* $\Lambda(2)$-*space* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for arbitrary $f \in H$ there is $g \in H^\perp$ satisfying the conditions $f + g \in L_\infty$, $\|f + g\|_\infty \leqslant \delta \|f\|_2$, and $\|g\|_2 \leqslant \varepsilon \|f\|_2$. In particular, a subset E of $\mathbb{Z}$ is said to be *uniformizable* $\Lambda(2)$*-set* if the corresponding subspace L_E of L_2 is a uniformizable $\Lambda(2)$-space.

In [8], Fournier proved that $E \subset \mathbb{Z}$ is a uniformizable $\Lambda(2)$-set if and only if the unit ball of L_E has equi-absolutely continuous norms in L_2. In the following statement we extend this result to arbitrary $\Lambda(2)$-spaces.

Theorem 5. *Let H be a closed subspace of L_2. The following conditions are equivalent:*

(a) *the set $B_H = \{f \in H\colon\ \|f\|_2 \leqslant 1\}$ has equi-absolutely continuous norms in L_2;*
(b) *H is a uniformizable $\Lambda(2)$-space.*

Proof. (b) $\to$ (a). By hypothesis, for every $\varepsilon > 0$ we can find $\delta > 0$ such that for arbitrary $f \in H$, $\|f\|_2 \leqslant 1$, there is $g \in H^\perp$ with the properties $\|f + g\|_\infty \leqslant \delta$ and $\|g\|_2 \leqslant \varepsilon$. Let $M > 2\delta$. From the embeddings

$$\bigl\{|f| > M\bigr\} \subset \bigl\{|f + g| + |g| > M\bigr\} \subset \bigl\{|g| > M - \delta\bigr\},$$

it follows that

$$\begin{aligned}\|f\chi_{\{|f|>M\}}\|_2 &\leqslant \|f\chi_{\{|g|>M-\delta\}}\|_2 \leqslant \bigl\|(f+g)\chi_{\{|g|>M-\delta\}}\bigr\|_2 \\ &\quad + \|g\chi_{\{|g|>M-\delta\}}\|_2 \leqslant \delta\|\chi_{\{|g|>M-\delta\}}\|_2 + \|g\|_2 \\ &\leqslant \delta\bigl(m\bigl\{|g| > M-\delta\bigr\}\bigr)^{1/2} + \varepsilon \leqslant \frac{\delta}{M-\delta}\|g\|_2 + \varepsilon \leqslant 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the condition (a) follows from Lemma 1.

(a) $\to$ (b). Let $f \in H$ and $\varepsilon > 0$. Applying hypothesis and Lemma 1 once more, we find $M = M(\varepsilon) > 0$, not depending on f, such that

$$\|f\chi_{\{|f|>M\|f\|_2\}}\|_2 \leqslant \varepsilon\|f\|_2. \tag{25}$$

Denoting

$$f_1 := f, \qquad h_1 := f_1\chi_{\{|f_1|\leqslant M\|f_1\|_2\}}, \quad \text{and} \quad \varphi_1 := -f\chi_{\{|f_1|>M\|f_1\|_2\}},$$

we have

$$h_1 = f_1 + \varphi_1, \qquad \|h_1\|_\infty \leqslant M\|f\|_2, \quad \text{and} \quad \|\varphi_1\|_2 \leqslant \varepsilon\|f\|_2.$$

If $f_2 := \pi_H(\varphi_1)$, where $\pi_H : L_2 \to H$ is the orthogonal projection onto H, then $\|f_2\|_2 \leqslant \|\varphi_1\|_2 \leqslant \varepsilon\|f\|_2$. Since $f_2 \in H$, by (25), we have

$$\|f_2\chi_{\{|f_2|>M\|f_2\|_2\}}\|_2 \leqslant \varepsilon\|f_2\|_2.$$

Therefore, setting

$$h_2 := f_2\chi_{\{|f_2|\leqslant M\|f_2\|_2\}} \quad \text{and} \quad \varphi_2 := -f_2\chi_{\{|f_2|>M\|f_2\|_2\}},$$

we obtain

$$h_2 = f_2 + \varphi_2, \qquad \|h_2\|_\infty \leqslant M\|f_2\|_2 \leqslant M\varepsilon\|f\|_2 \quad \text{and} \quad \|\varphi_2\|_2 \leqslant \varepsilon\|f_2\|_2 \leqslant \varepsilon^2\|f\|_2.$$

Arguing as above, for every $n \in \mathbb{N}$ we find functions f_n, h_n, and φ_n $(n = 1, 2, \ldots)$ such that

$$\begin{aligned} &h_n = f_n + \varphi_n, \qquad f_{n+1} = \pi_H(\varphi_n), \qquad \|h_n\|_\infty \leqslant M\varepsilon^{n-1}\|f\|_2, \quad \text{and} \\ &\|\varphi_n\|_2 \leqslant \varepsilon^n\|f\|_2. \end{aligned} \tag{26}$$

Since $h_j = f_j + \pi_H(\varphi_j) + \pi_{H^\perp}(\varphi_j) = f_j + f_{j+1} + \pi_{H^\perp}(\varphi_j)$ $(j = 1, 2, \ldots)$, where $\pi_{H^\perp} : L_2 \to H^\perp$ is the orthogonal projection onto $H^\perp$, we get

$$\begin{aligned} \sum_{j=1}^n (-1)^{j-1}h_j &= \sum_{j=1}^n (-1)^{j-1}f_j + \sum_{j=1}^{n-1} (-1)^{j-1}f_{j+1} \\ &\quad + (-1)^{n-1}\pi_H(\varphi_n) + \sum_{j=1}^n (-1)^{j-1}\pi_{H^\perp}(\varphi_j) \\ &= f + (-1)^{n-1}\pi_H(\varphi_n) + \sum_{j=1}^n (-1)^{j-1}\pi_{H^\perp}(\varphi_j). \end{aligned}$$

Letting $n \to \infty$ and taking into account (26), from the last equality we obtain

$$\sum_{j=1}^\infty (-1)^{j-1}h_j = f + \sum_{j=1}^\infty (-1)^{j-1}\pi_{H^\perp}(\varphi_j),$$

where the series in the left-hand side (resp. in the right-hand side) converges in L_∞ (resp. L_2). Since $\|\pi_{H^\perp}(\varphi_j)\|_2 \leqslant \|\varphi_j\|_2 \leqslant \varepsilon^j\|f\|_2$, the function

$$g := \sum_{j=1}^\infty (-1)^{j-1}\pi_{H^\perp}(\varphi_j)$$

belongs to $H^\perp$, $\|g\|_2 \leqslant \varepsilon/(1-\varepsilon)\|f\|_2$, and

$$\|f+g\|_\infty = \left\|\sum_{j=1}^\infty (-1)^{j-1}h_j\right\|_\infty \leqslant \frac{M}{1-\varepsilon}\|f\|_2.$$

Obviously, this concludes the proof. $\square$

The question whether the orthogonal sum of two $\Lambda(2)$-sets is a $\Lambda(2)$-set as well is still open (the so-called $\Lambda(2)$-set union problem). At the same time, in [17], Kashin constructed an orthogonal decomposition of L_2 as a sum of two $\Lambda(2)$-spaces (see also [26, Theorem 8.22]). Here, we present a more elementary example of such a sort based on using independent functions.

Theorem 6. *There exist mutually orthogonal $\Lambda(2)$-spaces H_1 and H_2 such that the sum $H_1 \oplus H_2$ is not a $\Lambda(2)$-space.*

Proof. For every $k = 1, 2, \ldots$ we set

$$u_k := 2^{k/2}\left(\chi_{[0,2^{-k-2}]} - \chi_{(2^{-k-2},2^{-k-1}]}\right), \qquad v_k := \chi_{(2^{-k-1},1/2+2^{-k-2}]} - \chi_{(1/2+2^{-k-2},1]},$$

and

$$h_{k,1} := u_k + c_k v_k, \qquad h_{k,2} := u_k - c_k v_k,$$

where reals c_k are chosen to satisfy the orthogonality condition $\int_0^1 h_{k,1}(s)h_{k,2}(s)\,ds = 0$ $(k = 1, 2, \ldots)$. Since

$$\int_0^1 h_{k,1}(s)h_{k,2}(s)\,ds = 2^{-k-1} \cdot 2^k - c_k^2\left(1 - 2^{-k-1}\right) = \frac{1}{2} - c_k^2\left(1 - 2^{-k-1}\right),$$

we set $c_k = (2(1 - 2^{-k-1}))^{-1/2}$.

It is well-known that there is a measure-preserving mapping from $([0,1], m)$ onto the product measure space $([0,1]^\infty, \prod_{k=1}^\infty m_k)$ (here, m_k is the usual Lebesgue measure). Therefore, rather than $L_2[0,1]$ we may consider the space $\mathbf{L}_2 := L_2([0,1]^\infty, \prod_{k=1}^\infty m_k)$. We set $f_k(\bar{t}) := h_{k,1}(t_k)$ and $g_k(\bar{t}) := h_{k,2}(t_k)$ $(k = 1, 2, \ldots)$, where $\bar{t} = (t_1, t_2, \ldots) \in [0,1]^\infty$. Let H_1 (resp. H_2) be the subspace spanned by f_k's (resp. g_k's) in $\mathbf{L}_2$. First, by definition of the functions $h_{k,1}$ and $h_{k,2}$, we have

$$\int_{[0,1]^\infty} f_k(\bar{t})g_j(\bar{t})\,d\bar{t} = \int_0^1 h_{k,1}(t_k)\,dt_k \int_0^1 h_{j,2}(t_j)\,dt_j = 0 \quad (1 \leqslant k \neq j < \infty).$$

Moreover, thanks to the choice of c_k,

$$\int_{[0,1]^\infty} f_k(\bar{t})g_k(\bar{t})\,d\bar{t} = \int_0^1 h_{k,1}(t_k)h_{k,2}(t_k)\,dt_k = 0 \quad (k = 1, 2, \ldots).$$

Therefore, the subspaces H_1 and H_2 are mutually orthogonal. Moreover, precisely in the same way as in Example 2, we can check that H_1 and H_2 are $\Lambda(2)$-spaces.

At the same time, it is easy to see that the sum $H_1 \oplus H_2$ is not a $\Lambda(2)$-space. Indeed, it suffices to note that $f_k + g_k = 2u_k \in H_1 + H_2$ for all $k = 1, 2, \ldots$, $\|f_k + g_k\|_2 = 1$, and $\|f_k + g_k\|_1 = 2 \cdot 2^{k/2} 2^{-k-1} = 2^{-k/2}$, $k = 1, 2, \ldots$. $\square$

As is well known, under the extra condition that at least one of two $\Lambda(2)$-sets is uniformizable, their orthogonal sum is a $\Lambda(2)$-set as well (see e.g. [8]). In conclusion, we extend this result to general $\Lambda(2)$-spaces.

Theorem 7. *Let H_1 (resp. H_2) be a uniformizable $\Lambda(2)$-space (resp. $\Lambda(2)$-space). If H_1 and H_2 are mutually orthogonal, then their direct sum $H_1 \oplus H_2$ is a $\Lambda(2)$-space.*

Proof. Assuming the contrary, we find $f_k \in H_1$ and $g_k \in H_2$ ($k = 1, 2, \ldots$) such that $\|f_k + g_k\|_2 = 1$, $k = 1, 2, \ldots$, and $f_k + g_k \to 0$ in measure. First, suppose that there is $\delta > 0$ such that $f_k + g_k \in M_{L_2,\delta}$ for all $k = 1, 2, \ldots$. Then, as is noted before Lemma 5, we have

$$\|f_k + g_k\|_1 \asymp 1, \quad k = 1, 2, \ldots \tag{27}$$

On the other hand, passing to a subsequence if it is necessary we can assume that $f_k + g_k \to 0$ a.e. Moreover, denoting $h_k = f_k + g_k$, for each $M > 0$ we have

$$\begin{aligned}
\int_0^1 |h_k|\,dt &= \int_{\{|h_k| \geqslant M\}} |h_k|\,dt + \int_{\{|h_k| < M\}} |h_k|\,dt \\
&\leqslant M^{-1} \int_{\{|h_k| \geqslant M\}} h_k^2\,dt + \int_{\{|h_k| < M\}} |h_k|\,dt \\
&\leqslant M^{-1} + \int_{\{|h_k| < M\}} |h_k|\,dt.
\end{aligned}$$

For any $\varepsilon > 0$, by the Lebesgue Bounded Convergence Theorem, there is a positive integer N such that

$$\int_{\{|h_k| < M\}} |h_k|\,dt < \frac{\varepsilon}{2} \quad \text{for all } k \geqslant N.$$

Thus, setting $M = 2/\varepsilon$ we obtain $\int_0^1 |f_k + g_k|\,dt \leqslant \varepsilon$ for $k \geqslant N$, which contradicts (27) if ε is sufficiently small.

Therefore, the sequence $\{f_k + g_k\}_{k=1}^\infty$ is not contained in $M_{L_2,\delta}$ for any $\delta > 0$, and applying Lemma 5 we can select a subsequence (which will be denoted in the same way) and a sequence $\{A_k\}$ of pairwise disjoint subsets of $[0, 1]$ such that

$$\lim_{k\to\infty} \left\| f_k + g_k - (f_k + g_k)\chi_{A_k} \right\|_2 = 0. \tag{28}$$

Moreover, since H_1 is a uniformizable $\Lambda(2)$-space and $\|f_k\|_2 \leqslant \|f_k + g_k\|_2 = 1$, by Theorem 5, we have

$$\lim_{k\to\infty} \|f_k\chi_{A_k}\|_2 = 0. \tag{29}$$

On the other hand, taking into account that H_1 and H_2 are mutually orthogonal, we obtain

$$\begin{aligned}\left\| f_k + g_k - (f_k + g_k)\chi_{A_k} \right\|_2^2 &= \|f_k - f_k\chi_{A_k}\|_2^2 + \|g_k - g_k\chi_{A_k}\|_2^2 \\ &\quad + (f_k - f_k\chi_{A_k}, g_k - g_k\chi_{A_k}) + (g_k - g_k\chi_{A_k}, f_k - f_k\chi_{A_k}) \\ &= \|f_k - f_k\chi_{A_k}\|_2^2 + \|g_k - g_k\chi_{A_k}\|_2^2 \\ &\quad - (f_k\chi_{A_k}, g_k) - (g_k, f_k\chi_{A_k}).\end{aligned}$$

By the Cauchy–Schwarz–Bunyakovskii inequality, the fact that $\|g_k\|_2 \leqslant 1$, and (29), we have

$$\lim_{k\to\infty} (f_k\chi_{A_k}, g_k) = \lim_{k\to\infty} (g_k, f_k\chi_{A_k}) = 0.$$

Therefore, from the preceding equality and (28) it follows that $\|f_k - f_k\chi_{A_k}\|_2 \to 0$. Applying (29) once more, we obtain $\|f_k\|_2 \to 0$. Combining this with the fact that $f_k + g_k \to 0$ in measure we infer that $g_k \to 0$ in measure. Since H_2 is a $\Lambda(2)$-space, we obtain $\|g_k\|_2 \to 0$, and hence $\|f_k + g_k\|_2 \to 0$, which contradicts the assumption. □

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