
A weighted Khintchine inequality

Sergey V. Astashkin and Guillermo P. Curbera

Abstract. We prove a weighted version of the well-known Khintchine inequality for rearrangement invariant norms.

1. Introduction and main results

The classical Khintchine inequality states, for $0 < p < \infty$, that there exist constants $A_p, B_p > 0$ such that

$$A_p \left(\sum_{i=1}^{\infty} a_i^2 \right)^{1/2} \leq \left(\int_0^1 \left| \sum_{i=1}^{\infty} a_i r_i(t) \right|^p dt \right)^{1/p} \leq B_p \left(\sum_{i=1}^{\infty} a_i^2 \right)^{1/2},$$

for every $(a_i) \in \ell^2$, where (r_i) are the Rademacher functions, that is, $r_i(t) := \text{sign} \sin(2^i \pi t)$, $t \in [0, 1]$, $i \in \mathbb{N}$.

A weighted version of the above inequality was recently proved in [18]. Namely, let w be a weight satisfying the following conditions

- (a) for some $q > p$ we have $w \in L^q([0, 1])$;
- (b) the support of w satisfies $m(\text{supp}(w)) > 2/3$.

Then there exist constants $C_1, C_2 > 0$, depending on p and w , such that for every $a = (a_i) \in \ell^2$

$$(1.1) \quad C_1^{-1} \|a\|_2 \leq \left(\int_0^1 \left| \sum_{i=1}^{\infty} a_i r_i(t) \right|^p |w(t)|^p dt \right)^{1/p} \leq C_2 \|a\|_2,$$

where $\|a\|_2 := \left(\sum_{i=1}^{\infty} a_i^2 \right)^{1/2}$.

In this paper we will consider the extension of the above inequality in two directions. On the one hand, instead of the family of L^p -spaces we consider the inequality for the essentially larger family of rearrangement invariant spaces. On

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the other hand, we look into the restriction on the measure of the support of the weight. Note that some restriction on the support of the weight is needed in order to have the lower estimate in (1.1) because there are Rademacher series with large zero set; see Proposition 7. In this regard it is noteworthy the following result, due to Stechkin and Ul'yanov, [16], on sets of uniqueness for Rademacher series: if $g = \sum_{i=1}^{\infty} a_i r_i$ and $m(\text{supp}(g)) < 1/2$ then $a_i = 0$ for all $i \geq 1$. The constant $1/2$ is sharp, since for $g = r_1 + r_2$ we have $m(\text{supp}(g)) = 1/2$. We replace the condition on the size of the support of w by a condition that depends on the structure of the support. A lower estimate in the weighted Khintchine inequality for weights having support with arbitrarily small measure is then possible.

Consider the following class of subsets of $[0, 1]$. By Δ_k^n we will denote the dyadic intervals of order n , that is, $\Delta_k^n = [(k-1)/2^n, k/2^n]$, for $n \in \mathbb{N}$ and $k = 1, 2, \dots, 2^n$. We say that a measurable set $E \subset [0, 1]$ belongs to the class $\mathcal{E}$ if there exist $n \in \mathbb{N}$, $\varepsilon \in (0, 2^{-n-2})$, $\delta \in (0, 1)$, and $\gamma \in (1/2, 1)$ such that the following conditions are satisfied:

- (i) there is a set $I \subset \{1, \dots, 2^n\}$, $\text{card } I > \gamma 2^n$, such that, for every $k \in I$, the set $E \cap \Delta_k^n$ is symmetric with respect to the midpoint of the interval Δ_k^n and $m(E \cap \Delta_k^n) > \delta$.
- (ii) there exists k_0 , $1 \leq k_0 \leq 2^n$, such that $m(E \cap \Delta_{k_0}^n) > \varepsilon + 3 \cdot 2^{-n-2}$.

We will denote this situation by $E \in \mathcal{E}_{\varepsilon, \delta, \gamma}^n$.

Let $\Lambda(X)$ be the Rademacher multiplier space of a rearrangement invariant (r.i.) space X (for the definitions and discussion related to basic concepts, see below). Moreover, for any $w \in \Lambda(X)$ and $\eta > 0$ we denote

$$M_\eta(w) := \{t \in [0, 1] : |w(t)| \geq \eta \|w\|_{\Lambda(X)}\}.$$

The main result of this paper is the following weighted version of Khintchine inequality.

Theorem 1. *Let X be an r.i. space on $[0, 1]$ such that the Rademacher functions generate in X a subspace isomorphic to ℓ^2 . Let $w \in \Lambda(X)$ be such that there exists $\eta > 0$ satisfying at least one of the following conditions:*

$$(a) \quad \alpha_\eta := \max \left\{ m(M_\eta(w) \cap [0, \frac{1}{2}]), m(M_\eta(w) \cap [\frac{1}{2}, 1]) \right\} > \frac{1}{4};$$

(b) the set $M_\eta(w)$ contains a set $E \in \mathcal{E}$.

Then, for every $a = (a_i) \in \ell^2$,

$$(1.2) \quad D_w \|w\|_{\Lambda(X)} \|a\|_2 \leq \left\| w \cdot \sum_{i=1}^{\infty} a_i r_i \right\|_X \leq C_X \|w\|_{\Lambda(X)} \|a\|_2,$$

where C_X depends only on X , and, in the case (a), $D_w = \frac{\eta}{4}(\alpha_\eta - \frac{1}{4})^2$, and, in the case (b), $D_w = \frac{\eta}{32} \min \{2\varepsilon, \delta(\gamma - \frac{1}{2})\} \cdot \min \{2^{n-1}\varepsilon, \frac{1}{4}(\gamma - \frac{1}{2})\}$, where $E \in \mathcal{E}_{\varepsilon, \delta, \gamma}^n$, with $n \in \mathbb{N}$, $\varepsilon \in (0, 2^{-n-2})$, $\delta \in (0, 1)$, and $\gamma \in (1/2, 1)$.

Corollary 2. *Let X be an r.i. space on $[0, 1]$ such that the Rademacher functions generate in X a subspace isomorphic to ℓ^2 . Let $w \in \Lambda(X)$ be such that*

$$\alpha_w := \max \left\{ m(\text{supp}(w) \cap [0, \frac{1}{2}]), m(\text{supp}(w) \cap [\frac{1}{2}, 1]) \right\} > \frac{1}{4}.$$

Then, for all sufficiently small $\eta > 0$, inequality (1.2) holds with $D_w = \frac{\eta}{4}(\alpha_w - \frac{1}{4})^2$.

Regarding the concepts appearing in Theorem 1 and Corollary 2, recall that the distribution function of a measurable function f is $m_f(\tau) := m(\{t \in [0, 1] : |f(t)| > \tau\})$, where m is the Lebesgue measure on $[0, 1]$. An rearrangement invariant (r.i.) space X is a Banach space of classes of measurable functions on $[0, 1]$ such that if $m_g(\tau) \leq m_f(\tau)$, for all $\tau > 0$, and $f \in X$ then $g \in X$ and $\|g\|_X \leq \|f\|_X$. For normalization purposes we will assume that $\|\chi_{[0, 1]}\|_X = 1$. The class of r.i. spaces contains many well-known families of function spaces: L^p , Orlicz, $L^{p, q}$, Lorentz, Marcinkiewicz, Zygmund, and many other. For r.i. spaces, see [6], [12].

Let $\mathcal{R}$ denote the set of all functions of the form $\sum_{i=1}^{\infty} a_i r_i$, where the series converges a.e., that is, $(a_i) \in \ell^2$, see Theorem V.8.2 in [19]. For an r.i. space X , let $\mathcal{R}(X)$ be the closed linear subspace of X given by $\mathcal{R} \cap X$. The Khintchine inequality shows, for $X = L^p$, $1 \leq p < \infty$, that $\mathcal{R}(X)$ is isomorphic to ℓ^2 . If $X = L^\infty$, then $\mathcal{R}(X) = \ell^1$. A result of Rodin and Semenov characterizes when $\mathcal{R}(X) \approx \ell^2$. Let L_N be the Orlicz space associated to the function $N(t) := \exp(t^2) - 1$, and $(L_N)_0$ be the closure of L^∞ in L_N . Then $\mathcal{R}(X) \approx \ell^2$, that is,

$$(1.3) \quad c_X \|(a_i)\|_2 \leq \left\| \sum_{i=1}^{\infty} a_i r_i \right\|_X \leq C_X \|(a_i)\|_2,$$

for some constants $C_X, c_X > 0$, if and only if $(L_N)_0 \subset X$, [13].

If X is an r.i. space on $[0, 1]$, the Rademacher multiplier space $\Lambda(X)$ consists of all measurable functions $f: [0, 1] \rightarrow \mathbb{R}$ such that $f \sum_{i=1}^{\infty} a_i r_i \in X$, for every $\sum_{i=1}^{\infty} a_i r_i \in \mathcal{R}(X)$. It is a Banach function space on $[0, 1]$ when endowed with the norm

$$(1.4) \quad \|f\|_{\Lambda(X)} := \sup \left\{ \left\| f \sum_{i=1}^{\infty} a_i r_i \right\|_X : \sum_{i=1}^{\infty} a_i r_i \in X, \left\| \sum_{i=1}^{\infty} a_i r_i \right\|_X \leq 1 \right\}.$$

Remarks 3. (a) The space $\Lambda(X)$ need not be r.i. However, in many cases it is possible to identify the largest r.i. space embedded into $\Lambda(X)$ (its symmetric kernel) denoted by $\text{Sym}(X)$. For example, $\text{Sym}(L^p)$ is the Zygmund space $L^p(\log L)^{1/2}$, $1 \leq p < \infty$. For more facts on $\Lambda(X)$ and $\text{Sym}(X)$, see [1]–[5], [10], [11].

(b) For the case $X = L^p$, $1 \leq p < \infty$, the condition $w \in \Lambda(L^p)$ in Theorem 1 is more general than $w \in L^q$ for some $q > p$, in [18]. To see this take into account that $L^q \subseteq L^p(\log L)^{1/2} \subseteq \Lambda(L^p)$.

(c) Note that all sets satisfying the condition $m(\text{supp}(w)) > 2/3$, which is used in [18], and even the weaker one $m(\text{supp}(w)) > 1/2$, satisfy also condition (a) of Theorem 1. Condition (b) in Theorem 1 on a weight w depends not so much on the size of the support of w as on its structure, showing that the lower estimate in the

weighted Khintchine inequality can hold for weights having support with arbitrarily small measure. It is instructive to emphasize that either of the conditions (a) and (b) guarantees that for every Rademacher zero set F (this means that there is a Rademacher sum $g = \sum_{i=1}^{\infty} a_i r_i$ vanishing on F) we have: $\text{supp}(g) \setminus F \neq \emptyset$ (regarding Rademacher zero sets see also the Appendix).

2. Proofs

The proof of the upper bound in Theorem 1 follows directly from the definition of the Rademacher multiplier space. Indeed, since by hypothesis we have (1.3), then from (1.4) it follows that

$$\sup_{a \in \ell^2, a \neq 0} \frac{\|w \cdot \sum_{i=1}^{\infty} a_i r_i\|_X}{\left(\sum_{i=1}^{\infty} a_i^2\right)^{1/2}} \leq C_X \cdot \sup_{a \in \ell^2, a \neq 0} \frac{\|w \cdot \sum_{i=1}^{\infty} a_i r_i\|_X}{\left\|\sum_{i=1}^{\infty} a_i r_i\right\|_X} = C_X \|w\|_{\Lambda(X)}.$$

Note that the definition of the Rademacher multiplier space $\Lambda(X)$ also shows that this upper bound is (up to a constant) optimal.

To prove the lower bound we will need the following assertion.

Proposition 4. *Let $E \subset [0, 1]$ be a measurable set.*

(a) *Suppose that E satisfies*

$$\alpha := \max \left\{ m(E \cap [0, \frac{1}{2}]), m(E \cap [\frac{1}{2}, 1]) \right\} > \frac{1}{4}.$$

Then for $0 < B \leq \frac{1}{2}(\alpha - \frac{1}{4})$ and for every $a = (a_i) \in \ell^2$ we have

$$m\left(\left\{t \in E : \left|\sum_{i=1}^{\infty} a_i r_i(t)\right| \geq B\|a\|_2\right\}\right) > \frac{1}{2}\left(\alpha - \frac{1}{4}\right).$$

(b) *Suppose that $E \in \mathcal{E}$. Then, for every $a = (a_i) \in \ell^2$, we have*

$$m\left(\left\{t \in E : \left|\sum_{i=1}^{\infty} a_i r_i(t)\right| \geq B\|a\|_2\right\}\right) > \frac{1}{4} \min \left\{ 2\varepsilon, \delta(\gamma - \frac{1}{2}) \right\},$$

for $0 < B \leq B_{n,\varepsilon,\gamma} := \min(2^{n-1}\varepsilon, \frac{1}{4}(\gamma - \frac{1}{2}))$, where $E \in \mathcal{E}_{\varepsilon,\delta,\gamma}^n$ for $n \in \mathbb{N}$, $\varepsilon \in (0, 2^{-n-2})$, $\delta \in (0, 1)$, and $\gamma \in (1/2, 1)$.

Proof. We will apply two well-known results. First, the Paley-Zygmund inequality (see, for instance, Lemma V.8.26 in [19] or Lemma 1 in [7]) will be used in the following form: for any $A > B \geq 0$ and for arbitrary non-negative random variable f with $\mathbb{E}f \geq A$ and $\mathbb{E}f^2 = 1$ we have

$$m(f \geq B) \geq (A - B)^2.$$

Moreover, by [17], the best constant A_1 in Khintchine inequality for $p = 1$ is equal to $1/\sqrt{2}$. Therefore, for every $a = (a_i) \in \ell^2$

$$\int_0^1 \left| \sum_{i=1}^{\infty} a_i r_i(t) \right| dt \geq \frac{1}{\sqrt{2}} \|a\|_2,$$

and we can apply Paley-Zygmund inequality to the function $f := |\sum_{i=1}^{\infty} a_i r_i|/\|a\|_2$ with $A = 1/\sqrt{2}$. Then, for any B with $0 \leq B < 1/\sqrt{2}$, we have

$$(2.1) \quad m\left(\left\{t \in [0, 1] : \left| \sum_{i=1}^{\infty} a_i r_i(t) \right| \geq B\|a\|_2\right\}\right) \geq \left(\frac{1}{\sqrt{2}} - B\right)^2.$$

We first prove (a). Assume, for example, that $\alpha = m(E \cap [0, 1/2])$. Denote $Q_B := \{t \in [0, 1/2] : |\sum_{i=1}^{\infty} a_i r_i(t)| \geq B\|a\|_2\}$. Due to elementary properties of the Rademacher functions, the set $\{t \in [0, 1] : |\sum_{i=1}^{\infty} a_i r_i(t)| \geq B\|a\|_2\}$ is symmetric with respect to the point $1/2$ and thus from (2.1) it follows that

$$m(Q_B) \geq \frac{1}{2} \left(\frac{1}{\sqrt{2}} - B\right)^2.$$

Hence, provided that $0 < B \leq \frac{1}{2}(\alpha - \frac{1}{4})$ we have

$$m(E \cap Q_B) \geq m(E \cap [0, 1/2]) + m(Q_B) - \frac{1}{2} \geq \alpha + \frac{1}{2} \left(\frac{1}{\sqrt{2}} - B\right)^2 - \frac{1}{2} > \frac{1}{2} \left(\alpha - \frac{1}{4}\right),$$

and the inequality in case (a) is proved. Similar estimates establish the result when $\alpha = m(E \cap [1/2, 1])$.

Now, we proceed to the proof of (b). Assume that $E \in \mathcal{E}_{\varepsilon, \delta, \gamma}^n$, for some $n \in \mathbb{N}$, $\varepsilon > 0$, $\delta \in (0, 1)$, and $\gamma \in (1/2, 1)$. Let $g = \sum_{i=1}^{\infty} a_i r_i$ be an arbitrary Rademacher series, where $a = (a_i) \in \ell^2$, and let $Q_{E,B} := \{t \in E : |g(t)| \geq B\|a\|_2\}$.

Firstly, we suppose that

$$(2.2) \quad \sum_{i=1}^n a_i^2 \leq \sum_{i=n+1}^{\infty} a_i^2.$$

By hypothesis, there exists k_0 , $1 \leq k_0 \leq 2^n$, such that $m(E \cap \Delta_{k_0}^n) > \varepsilon + 3 \cdot 2^{-n-2}$. Since the function $\sum_{i=1}^n a_i r_i$ is constant on the interval $\Delta_{k_0}^n$, say a_0 , then, using (2.2), we obtain that

$$\begin{aligned} m(Q_{E,B}) &\geq m\left\{t \in E : |g(t)| \geq \sqrt{2}B \left(\sum_{i=n+1}^{\infty} a_i^2\right)^{1/2}\right\} \\ &\geq m\left\{t \in E \cap \Delta_{k_0}^n : \left|a_0 + \sum_{i=n+1}^{\infty} a_i r_i(t)\right| \geq \sqrt{2}B \left(\sum_{i=n+1}^{\infty} a_i^2\right)^{1/2}\right\}. \end{aligned}$$

Therefore, we can choose a set $F \subset [0, 1]$ with $m(F) = 2^n m(E \cap \Delta_{k_0}^n) > 3/4$ and such that

$$(2.3) \quad m(Q_{E,B}) \geq 2^{-n} m\left(\left\{t \in F : \left|a_0 + \sum_{i=n+1}^{\infty} a_i r_i(t)\right| \geq \sqrt{2}B \left(\sum_{i=n+1}^{\infty} a_i^2\right)^{1/2}\right\}\right).$$

At the same time, from (2.1) it follows that

$$m\left(\left\{t \in [0, 1] : \left| \sum_{i=n+1}^{\infty} a_i r_i(t) \right| \geq \sqrt{2}B \left(\sum_{i=n+1}^{\infty} a_i^2 \right)^{1/2} \right\}\right) \geq \left(\frac{1}{\sqrt{2}} - \sqrt{2}B \right)^2.$$

Combining this inequality with elementary symmetry properties of the Rademacher functions, we obtain that

$$m\left(\left\{t \in [0, 1] : \left| a_0 + \sum_{i=n+1}^{\infty} a_i r_i(t) \right| \geq \sqrt{2}B \left(\sum_{i=n+1}^{\infty} a_i^2 \right)^{1/2} \right\}\right) \geq \frac{1}{2} \left(\frac{1}{\sqrt{2}} - \sqrt{2}B \right)^2.$$

Therefore, arguing in the same way as in the proof of (a), we obtain that

$$\begin{aligned} m\left(\left\{t \in F : \left| a_0 + \sum_{i=n+1}^{\infty} a_i r_i(t) \right| \geq \sqrt{2}B \left(\sum_{i=n+1}^{\infty} a_i^2 \right)^{1/2} \right\}\right) \\ \geq m(F) + \frac{1}{2} \left(\frac{1}{\sqrt{2}} - \sqrt{2}B \right)^2 - 1 \\ \geq m(F) - \frac{3}{4} - B. \end{aligned}$$

Since

$$m(F) - \frac{3}{4} = 2^n (m(E \cap \Delta_{k_0}^n) - 3 \cdot 2^{-n-2}) > 2^n \varepsilon,$$

and, by hypotheses $B < 2^{n-1} \varepsilon$, from the previous inequality it follows that

$$m\left(\left\{t \in F : \left| a_0 + \sum_{i=n+1}^{\infty} a_i r_i(t) \right| \geq \sqrt{2}B \left(\sum_{i=n+1}^{\infty} a_i^2 \right)^{1/2} \right\}\right) \geq 2^{n-1} \varepsilon.$$

Hence, using (2.3), we obtain, provided $0 < B < 2^{n-1} \varepsilon$, that

$$(2.4) \quad m(Q_{E,B}) \geq \frac{\varepsilon}{2}.$$

Consider now the case when inequality (2.2) does not hold. Then

$$(2.5) \quad m(Q_{E,B}) \geq m\left\{t \in E : |g(t)| \geq \sqrt{2}B \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \right\}.$$

By hypothesis, if $I := \{k = 1, 2, \dots, 2^n : m(E \cap \Delta_k^n) > \delta\}$, then

$$(2.6) \quad \text{card } I > \gamma 2^n.$$

Again, by (2.1), for some $I' \subseteq \{1, 2, \dots, 2^n\}$, we have that

$$\begin{aligned} m\left(\left\{t \in [0, 1] : \left| \sum_{i=1}^n a_i r_i(t) \right| \geq \sqrt{2}B \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \right\}\right) \\ = \sum_{i \in I'} m(\Delta_i^n) = \text{card } I' \cdot 2^{-n} \geq \left(\frac{1}{\sqrt{2}} - \sqrt{2}B \right)^2, \end{aligned}$$

whence

$$\text{card } I' \geq 2^n \left(\frac{1}{\sqrt{2}} - \sqrt{2}B \right)^2.$$

It is easy to check that the condition $0 < B \leq \frac{1}{4}(\gamma - \frac{1}{2})$ guarantees that

$$\left(\frac{1}{\sqrt{2}} - \sqrt{2}B \right)^2 + \gamma - 1 > \frac{1}{2} \left(\gamma - \frac{1}{2} \right),$$

Therefore, by (2.6),

$$\begin{aligned} \text{card } (I' \cap I) &\geq \text{card } (I') + \text{card } (I) - 2^n > 2^n \left(\left(\frac{1}{\sqrt{2}} - \sqrt{2}B \right)^2 + \gamma - 1 \right) \\ (2.7) \qquad \qquad \qquad &> 2^{n-1} \left(\gamma - \frac{1}{2} \right). \end{aligned}$$

By hypothesis, for every $k \in I$, the set $E \cap \Delta_k^n$ is symmetric with respect to the midpoint of the interval Δ_k^n . Therefore, if $t \in E \cap \Delta_k^n$, where $k \in I \cap I'$ is fixed, then the symmetric point t' belongs also to the set $E \cap \Delta_k^n$. Note that the sum $\sum_{i=1}^n a_i r_i$ is constant on the interval $E \cap \Delta_k^n$, say b , and $|b| \geq \sqrt{2}B \left(\sum_{i=1}^n a_i^2 \right)^{1/2}$. On the other hand, since $r_i(t') = -r_i(t)$, if $i > n$, we have that $g(t) = b + c$ and $g(t') = b - c$ for some constant c . Obviously, then at least one of the inequalities

$$|g(t)| \geq \sqrt{2}B \left(\sum_{i=1}^n a_i^2 \right)^{1/2}, \quad |g(t')| \geq \sqrt{2}B \left(\sum_{i=1}^n a_i^2 \right)^{1/2}$$

holds. Thus, if $0 < B \leq \frac{1}{4}(\gamma - \frac{1}{2})$, by (2.5) and (2.7), we obtain that

$$\begin{aligned} m(Q_{E,B}) &\geq \sum_{k \in I \cap I'} m \left\{ t \in E \cap \Delta_k^n : |g(t)| \geq \sqrt{2}B \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \right\} \\ &\geq \frac{1}{2} \sum_{k \in I \cap I'} m \left\{ t \in E \cap \Delta_k^n : \left| \sum_{i=1}^n a_i r_i(t) \right| \geq \sqrt{2}B \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \right\} \\ &\geq \delta \cdot 2^{-n-1} \text{card } (I \cap I') > \frac{\delta}{4} \left(\gamma - \frac{1}{2} \right). \end{aligned}$$

Combining the last estimate together with inequality (2.4), we obtain the result. $\square$

Corollary 5. *Let $E \subseteq [0, 1]$ be a measurable set such that at least one of the following conditions holds:*

$$(a) \quad \alpha := \max \left\{ m(E \cap [0, \frac{1}{2}]), m(E \cap [\frac{1}{2}, 1]) \right\} > \frac{1}{4};$$

$$(b) \quad E \in \mathcal{E}.$$

Then, for every $a = (a_i) \in \ell^2$, we have that

$$\left\| \sum_{i=1}^{\infty} a_i r_i \chi_E \right\|_1 \geq d_E \|a\|_2,$$

where, in the case (a), we have $d_E = \frac{1}{4}(\alpha - \frac{1}{4})^2$, and, in the case (b), we have $d_E = \frac{1}{32} \min \{2\varepsilon, \delta(\gamma - \frac{1}{2})\} B_{n,\varepsilon,\gamma}$, where $E \in \mathcal{E}_{\varepsilon,\delta,\gamma}^n$, for $n \in \mathbb{N}$, $\varepsilon \in (0, 2^{-n-2})$, $\delta \in (0, 1)$, and $\gamma \in (1/2, 1)$.

Remark 6. Proposition 4 should be compared with the following inequality from the work of Burkholder (Theorem 1 in [7]): there exist universal constants $\xi_1 > 0$ and $\xi_2 > 0$ such that for every set $E \subset [0, 1]$ with $m(E) > 0$ there exists $N := N(E)$ with the following property:

$$m\left(\left\{t \in E : \left| \sum_{i=N}^{\infty} a_i r_i(t) \right| \geq \xi_2 \left(\sum_{i=N}^{\infty} a_i^2 \right)^{1/2} \right\}\right) \geq \xi_1 \cdot m(E),$$

for any $(a_i) \in \ell^2$. Of course, because of existing Rademacher zero sets (see Remark 3(c)), the latter inequality does not hold with $N = 1$ for all measurable $E \subset [0, 1]$. Nevertheless, Proposition 4 shows that there are some sets (including sets with arbitrarily small measure) for which an analogous estimate is obtained for the Rademacher series starting at $N = 1$. For issues regarding the local version of Khintchine inequality, see [5], [9], [14], [15].

Now, we prove the lower estimate in Theorem 1, for example, in the case when the condition (a) is satisfied. Let $\eta > 0$ be such that

$$\alpha_\eta := \max \left\{ m(M_\eta(w) \cap [0, \frac{1}{2}]), m(M_\eta(w) \cap [\frac{1}{2}, 1]) \right\} > \frac{1}{4}.$$

Since for an r.i. space X on $[0, 1]$ with $\|\chi_{[0,1]}\|_X = 1$ we have $\|x\|_1 \leq \|x\|_X$ ($x \in X$), then from Corollary 5 we deduce that

$$\begin{aligned} \left\| w \cdot \sum_{i=1}^{\infty} a_i r_i \right\|_X &\geq \left\| w \chi_{M_\eta(w)} \cdot \sum_{i=1}^{\infty} a_i r_i \right\|_X \\ &\geq \eta \|w\|_{\Lambda(X)} \left\| \chi_{M_\eta(w)} \cdot \sum_{i=1}^{\infty} a_i r_i \right\|_1 \\ &\geq \frac{\eta}{4} (\alpha_\eta - \frac{1}{4})^2 \|w\|_{\Lambda(X)} \|a\|_2, \end{aligned}$$

and the proof of the lower estimate in (1.2) is complete. In the case when the condition (b) holds this estimate can be proved by completely analogous arguments.

The proof of Corollary 2 follows from the equality $\text{supp}(w) = \cup_{\eta>0} M_\eta(w)$ and the steps of the proof of Proposition 4(a).

3. Appendix: Rademacher zero sets with measure 1/2

Proposition 7. *Let $g = \sum_{k=1}^{\infty} a_k r_k$. If*

$$m(\{t \in [0, 1] : g(t) = 0\}) = \frac{1}{2},$$

then there exist $1 \leq m < n$ and $a \in \mathbb{R}$ such that $g = a(r_n + r_m)$ or $g = a(r_n - r_m)$.

Proof. Since infinite Rademacher series are almost everywhere non-null (by Theorem 4 in [16], see also Corollary 1 in [7], and Corollary in [8]), there is $n \in \mathbb{N}$ such that $g = \sum_{k=1}^n a_k r_k$. Let $b_k := -a_k/a_n$ if $1 \leq k < n$. By considering $h = r_n - \sum_{k=1}^m b_k r_k$, where $b_m \neq 0$ and $1 \leq m < n$, from our hypothesis it follows that there exist a set $E \subset [0, 1]$ with $m(E) = 1/2$ such that

$$(3.1) \quad r_n(t) = \sum_{k=1}^m b_k r_k(t) \quad (t \in E).$$

We have $E = \cup_{i \in I} \Delta_i^n$ for some set $I \subset \{1, 2, \dots, 2^n\}$. For any $i \in I$ there is $j_i \in \{1, 2, \dots, 2^m\}$ such that $\Delta_i^n \subset \Delta_{j_i}^m$. Since the sum from the right side of equality (3.1) is constant on the interval $\Delta_{j_i}^m$, then it is equal to $+1$ or -1 on $\Delta_{j_i}^m$, depending on the value of r_n on the interval Δ_i^n . Therefore, on half of $\Delta_{j_i}^m$ we have $r_n(t) = \sum_{k=1}^m b_k r_k(t)$ and on the other half $-r_n(t) = \sum_{k=1}^m b_k r_k(t)$. Since $m(E) = 1/2$, it follows that $\{j_i : i \in I\} = \{1, 2, \dots, 2^m\}$. Consequently $|\sum_{k=1}^m b_k r_k(t)| = 1$, for all $t \in [0, 1]$, and, by Corollary 3 in [16], we have

$$m\left(\left\{t \in [0, 1] : \sum_{k=1}^m b_k r_k(t) = 1\right\}\right) = m\left(\left\{t \in [0, 1] : \sum_{k=1}^m b_k r_k(t) = -1\right\}\right) = \frac{1}{2}.$$

Moreover, assuming that $\sum_{k=1}^m b_k r_k(t) = 1$ if $t \in \Delta_i^m$ for some $1 \leq i < 2^m$, we see that on the next interval Δ_{i+1}^m only the function $r_m(t)$ changes its value. Hence, $\sum_{k=1}^m b_k r_k(t) = -1$ if $t \in \Delta_{i+1}^m$. Therefore, the sum $\sum_{k=1}^m b_k r_k(t)$ takes on intervals Δ_i^m alternately its values ± 1 . Thus, this sum coincides either with r_m or with $-r_m$, and the proof is complete. $\square$

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SERGEY V. ASTASHKIN: Department of Mathematics and Mechanics, Samara State University, ul. Akad. Pavlova 1, 443011 Samara, Russia

E-mail: astashkn@ssu.samara.ru

GUILLERMO P. CURBERA: Facultad de Matemáticas, Universidad de Sevilla, Aptdo. 1160, Sevilla 41080, Spain

E-mail: curbera@us.es

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