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BRIEF COMMUNICATIONS

On Complementability of Subspaces in Symmetric Spaces with the Kruglov Property*

S. V. Astashkin

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Abstract. We show that, for a broad class of symmetric spaces on $[0,1]$, the complementability of the subspace generated by independent functions f_k $(k=1,2,\dots)$ is equivalent to the complementability of the subspace generated by the disjoint translates $\bar f_k(t)=f_k(t-k+1)\chi_{[k-1,k)}(t)$ of these functions in some symmetric space Z_X^2 on the semiaxis $[0,\infty)$. Moreover, if $\sum_{k=1}^{\infty} m(\operatorname{supp} f_k) \leqslant 1$, then Z_X^2 can be replaced by X itself. This result is new even in the case of L_p-spaces. A series of consequences is obtained; in particular, for the class of symmetric spaces, a result similar to a well-known theorem of Dor and Starbird on the complementability in $L_p[0,1]$ $(1\leqslant p<\infty)$ of the subspace $[f_k]$ generated by independent functions provided that it is isomorphic to the space l_p is obtained.

Key words: complemented subspace, independent functions, Rademacher functions, symmetric space, Kruglov property, Boyd indices, lower p-estimate.

1. Introduction. Properties of independent functions and probabilistic inequalities related to them are widely applied to studying the geometry of function spaces (see, e.g., [1]–[3]). In many cases, the most important problem is whether or not the subspace generated by a given system of functions is complemented (i.e., whether or not there exists a bounded linear projection onto this subspace).

The main result announced in this note is Theorem 1 asserting that, for a wide class of symmetric spaces (s.s.) X on $[0,1]$ which, together with their Köthe duals, have the so-called Kruglov property, the complementability of the subspace generated by independent functions f_k $(k=1,2,\dots)$ is equivalent to that of the subspace generated by their disjointly supported copies in the symmetric space Z_X^2 on the semiaxis $[0,\infty)$, which consists of all functions f satisfying the condition

$$\|f\|_{Z_X^2} := \|f^*\chi_{[0,1]}\|_X + \|f^*\chi_{[1,\infty)}\|_{L_2[1,\infty)} < \infty, \tag{1}$$

where f^* is the left-continuous nonincreasing rearrangement of $|f|$. If $\sum_{k=1}^{\infty} m(\operatorname{supp} f_k)\leqslant 1$, then Z_X^2 can be replaced by the space X itself in this assertion (see Corollary 1). Note that these results are new even in the case of L_p-spaces.

A close connection between properties of independent functions and those of their disjointly supported copies was noticed a long time ago, at least, after Rosenthal's remarkable 1970 paper [4] on isomorphic classification of subspaces of L_p-spaces. In 1989 Johnson and Schechtman [5] generalized the Rosenthal inequalities to the class of s.s. They established that, for an arbitrary s.s. X on $[0,1]$, there is a constant $\beta>0$ such that any sequence $\{f_k\}_{k=1}^{\infty}\subset X$ of independent functions with $\int_0^1 f_k(t)\,dt=0$ $(k=1,2,\dots)$ satisfies the inequality

$$\Big\|\sum_{k=1}^{n}\bar f_k\Big\|_{Z_X^2} \leqslant \beta\Big\|\sum_{k=1}^{n} f_k\Big\|_X \qquad (n\in\mathbb{N}). \tag{2}$$

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Moreover, if there exists a $p < \infty$ such that $X \supset L_p$, then, for some $\alpha > 0$ depending only on the space X, we have the opposite inequality

$$\Big\| \sum_{k=1}^{n} f_k \Big\|_X \leqslant \alpha \Big\| \sum_{k=1}^{n} \bar{f}_k \Big\|_{Z_X^2} \qquad (n \in \mathbb{N}). \tag{3}$$

More recently, the author, jointly with Sukochev, showed in [6, Theorem 3.1] (see also [7, Theorem 25]) that inequality (3) holds also under the less restrictive and already sharp condition that an s.s. X possesses the so-called Kruglov property, which was introduced by Braverman [8] on the basis of some probabilistic constructions due to Kruglov [9] (see the definition in the next section). For instance, (3) is fulfilled for the exponential Orlicz spaces L_{N_γ} with $N_\gamma(u) = e^{u^\gamma} - 1$ for $0 < \gamma \leqslant 1$, which play an important role in applications (clearly, these spaces do not satisfy the condition "$X \supset L_p$ for some $p < \infty$"). Inequalities (2) and (3) are the main tools in the proof of Theorems 1 and 2 (the latter concerns the complementability of subspaces generated by sequences of independent functions equivalent to the standard basis of the space l_p). Theorem 3 considers the important particular case of the Lorentz spaces $L_{p,q}$ ($1 < p < \infty$, $1 \leqslant q < \infty$). Finally, in the last section of the paper one more characterization of the Hilbert space L_2 in terms of the complementability of its subspaces is presented (Theorem 4).

2. Definitions and notation. Let $J = [0,1]$ or $J = [0,\infty)$, and let m be the Lebesgue measure on J. For a measurable function $x = x(t)$ on J and $\tau > 0$, we define the distribution function $n_x(\tau) := m\{t \in J : |x(t)| > \tau\}$. Two functions x and y are said to be *equimeasurable* if $n_x(\tau) = n_y(\tau)$ ($\tau > 0$). In particular, any measurable function x on J is equimeasurable with the left-continuous nonincreasing *rearrangement* $x^*(t) := \inf\{\tau \geqslant 0 : n_x(\tau) < t\}$ of the function $|x|$.

A Banach space X of real-valued Lebesgue measurable functions on the interval J is called a *symmetric* space (s.s.) if (1) the conditions $x \in X$ and $|y(t)| \leqslant |x(t)|$ a.e. imply that $y \in X$ and $\|y\|_X \leqslant \|x\|_X$; (2) the conditions that functions $x \in X$ and y are equimeasurable and $x \in X$ imply that $y \in X$ and $\|y\|_X = \|x\|_X$.

If X is an s.s. on J, then the *Köthe dual* space X' consists of all functions y such that $\|y\|_{X'} = \sup\big\{ \int_J x(t)y(t)\,dt : \|x\|_X \leqslant 1 \big\} < \infty$. The space X' is also symmetric; it is isometrically embedded in the usual dual space X^*, and $X' = X^*$ if and only if X is separable. An s.s. X is called *maximal* if from the conditions $x_n \in X$ ($n = 1,2,\dots$), $\sup_{n=1,2,\dots} \|x_n\|_X < \infty$, and $x_n \to x$ a.e. it follows that $x \in X$ and $\|x\|_X \leqslant \limsup_{n\to\infty} \|x_n\|_X$. In the sequel, following [1], we shall suppose that all s.s. considered here are separable or maximal. The most important examples of s.s. are the spaces $L_p(J)$ which are both a part of the family of Lorentz spaces $L_{p,q}(J)$ ($1 < p < \infty$, $1 \leqslant q \leqslant \infty$) and a particular case of Orlicz spaces (for details related to s.s. we refer the reader to [1], [10], and [11]).

Let f be a measurable function on $[0,1]$, and let $\pi(f)$ denote the function (random variable) $\sum_{i=1}^{N} f_i$, where the f_i are independent copies of f and N is an random variable that has a Poisson distribution with parameter 1 and is independent with respect to the sequence $\{f_i\}$. An s.s. X on $[0,1]$ is said to have the *Kruglov property* ($X \in \mathbb{K}$) if $\pi(f) \in X$ for arbitrary $f \in X$. Simplifying the situation, we can say that $X \in \mathbb{K}$ if it is sufficiently "far" from the space L_∞. In particular, this is so if $X \supset L_p$ for some $p < \infty$. For instance, $L_{p,q} \in \mathbb{K}$ for all $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. In addition, if L_{N_γ} is an exponential Orlicz space, then $L_{N_\gamma} \in \mathbb{K}$ if and only if $0 < \gamma \leqslant 1$ [8, p. 42]. About the Kruglov property and its applications to studying the geometry of s.s. see [8] and [7].

The set Z_X^2 defined by condition (1) (X is an s.s. on $[0,1]$) appeared for the first time in [2]. Since the functional $\|\cdot\|_{Z_X^2}$ is equivalent to an appropriate symmetric norm, Z_X^2 is an s.s. on $[0,\infty)$. For a given sequence $\{f_k\}_{k=1}^\infty$ of functions defined on $[0,1]$, let $\{\bar{f}_k\}_{k=1}^\infty$ be an arbitrary sequence of disjointly supported copies of the f_k (i.e., functions identically distributed with the f_k) on $[0,\infty)$, for instance, $\bar{f}_k(t) = f_k(t-k+1)\chi_{[k-1,k)}(t)$.

We say that an s.s. X satisfies a *lower p-estimate* ($1 \leqslant p < \infty$) if there is a constant $C_X > 0$ such that, for arbitrary disjointly supported functions $x_1, \dots, x_n$ from X, we have $\left(\sum_{k=1}^n \|x_k\|_X^p\right)^{1/p} \leqslant C_X \left\| \sum_{k=1}^n x_k \right\|_X$.

3. Main result.

Theorem 1. *Suppose that an s.s. X on $[0,1]$ satisfies the conditions $X \in \mathbb{K}$ and $X' \in \mathbb{K}$. For an arbitrary sequence $\{f_k\}_{k=1}^\infty \subset X$ of independent functions, the closed linear span $[f_k]$ is complemented in X if and only if the subspace $[\bar{f}_k]$ generated by disjointly supported copies of the functions f_k is complemented in the space Z_X^2.*

Remark 1. In the statement of Theorem 1 we took into account the fact that the complementability of the subspace $[\bar{f}_k]$ in Z_X^2 does not depend on the choice of the disjointly supported copies $\bar{f}_k$. Moreover, if functions g_k and h_k in an s.s. Z on the semiaxis $[0, \infty)$ are equimeasurable for every $k = 1, 2, \dots$, $\operatorname{supp} g_k \subset [k-1, k]$, and $\operatorname{supp} h_k \subset [k-1, k]$ ($k = 1, 2, \dots$), then the subspace $[g_k]$ is complemented in Z if and only if so is $[h_k]$.

Remark 2. In the general case, none of the conditions $X \in \mathbb{K}$ and $X' \in \mathbb{K}$ in Theorem 1 can be omitted. Indeed, the Rademacher functions $r_k(t) = \operatorname{sign} \sin 2^k \pi t$ ($k = 1, 2, \dots$) span a complemented subspace in an s.s. X if and only if $G \subset X \subset G'$, where G is the closure of the space L_∞ in the Orlicz space L_{N_2} (see [12] or [1, Theorem 2.b.4(ii)]). At the same time, the sequence of their disjointly supported copies $\{\chi_{[k-1,k-1/2]} - \chi_{[k-1/2,k]}\}$ (or, equivalently, $\{\chi_{[k-1,k]}\}$) spans a complemented subspace in Z_X^2 for every s.s. X [10, inequality (3.11) and Theorem 2.4.3].

Corollary 1. *Suppose that an s.s. X satisfies the conditions of Theorem* 1, *and let a sequence $\{f_k\} \subset X$ of independent functions be such that $\sum_{k=1}^\infty m(\operatorname{supp} f_k) \leqslant 1$. Then the subspace $[f_k]$ is complemented in X if and only if the subspace $[\bar{f}_k]$ is complemented in the same space.*

Proof. First, as is easily seen, X is complemented in Z_X^2. In addition, if $f \in Z_X^2$ is such that $m(\operatorname{supp} f) \leqslant 1$, then $\|f\|_{Z_X^2} = \|f\|_X$. Therefore, the complementability of the subspace $[\bar{f}_k]$ in Z_X^2 is equivalent to that in X. Applying Theorem 1, we obtain the required result. $\square$

4. On sequences of independent functions equivalent to the standard basis of l_p. Using Theorem 1, we can obtain the following result on the complementability of subspaces of the s.s. generated by sequences of independent functions that are equivalent to the standard basis in l_p.

Theorem 2. *Let $1 \leqslant p < \infty$. Suppose that an s.s. X satisfies a lower p-estimate, $X' \in \mathbb{K}$, and $\{f_k\} \subset X$ is a sequence of independent functions equivalent in X to the standard basis in l_p. Then, if either* (a) $p > 2$, (b) $p = 2$ *and* $X \supset L_2$, *or* (c) $\|f_k \chi_{A_k}\|_X \geqslant \delta > 0$ ($k = 1, 2, \dots$) *for some sequence of sets $A_k \subset [0,1]$ such that $\sum_{k=1}^\infty m(A_k) < \infty$, then the subspace $[f_k]$ is complemented in X.*

Remark 3. The almost disjointness of the sequence $\{f_k\} \subset X$ in condition (c) of Theorem 2 is essential. Indeed, it is well known that the sequence of independent p-stable random variables is equivalent in $L_r = L_r[0,1]$ to the standard basis of the space l_p if $1 \leqslant r < p < 2$; moreover, such a sequence generates an uncomplemented subspace in L_r (see, e.g., [13, Theorems 6.4.18 and 6.4.21]).

Now, let us consider the particular case of the Lorentz spaces $L_{p,q} = L_{p,q}[0,1]$. Since $L_{p,q}$ satisfies a lower $\max(p,q)$-estimate (see [14] or [15, Theorem 3]), $L_{p,q} \in \mathbb{K}$, and $(L_{p,q})' = L_{p',q'} \in \mathbb{K}$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, Theorem 2 implies the following assertion.

Theorem 3. *Let $1 < p \leqslant q < \infty$. Suppose that $\{f_k\} \subset L_{p,q}$ is a sequence of independent functions equivalent in $L_{p,q}$ to the standard basis in l_q. Then the subspace $[f_k]$ is complemented in $L_{p,q}$ whenever either $p \geqslant 2$ or $\|f_k \chi_{A_k}\|_{L_{p,q}} \geqslant \delta > 0$ ($k = 1, 2, \dots$) for some sequence of sets $A_k \subset [0,1]$ such that $\sum_{k=1}^\infty m(A_k) < \infty$.*

Suppose that $1 \leqslant p < \infty$ and a sequence $\{f_k\}_{k=1}^\infty$ of independent functions in $L_p = L_p[0,1]$ such that $\|f_k\|_p = 1$ and $\int_0^1 f_k(t)\,dt = 0$ ($k = 1, 2, \dots$) spans a subspace $[f_k]$ of L_p isomorphic to l_p. By using arguments from the articles [4] (see Theorem 4 and Lemma 7) and [16] (see the beginning of the proof of Theorem A), it is not hard to show that any such sequence $\{f_k\}$ is equivalent in L_p to the standard basis of the space l_p. In the case $p \neq 2$, the inequality $\|f_k \chi_{A_k}\|_{L_p} \geqslant \delta > 0$ ($k = 1, 2, \dots$) holds for some sequence of pairwise disjoint sets $A_k \subset [0,1]$ [17, Theorem B] and $L_{p,p} = L_p$ ($1 < p < \infty$); therefore, Theorems 2 and 3 imply the well-known Dor–Starbird theorem.

Corollary 2 [16, Theorem A]**.** *Let $1 \leqslant p < \infty$, and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of independent functions in $L_p = L_p[0,1]$. If the subspace $[f_k]$ is isomorphic to l_p, then it is complemented in L_p.*

5. Characterization of L_2. In [18] Hernandez and Semenov studied the complementability of the subspace of an s.s. X on $[0, \infty)$ generated by the translations $a_k = a(t - k + 1)\chi_{[k-1,k)}(t)$ ($k = 1, 2, \dots$) of a fixed function $a \in X$ such that $a = a^* \neq 0$ and $\operatorname{supp} a \subset [0,1]$. Theorem 1 in this note and Theorem 5.4 in [18] give the following characterization of the Hilbert space L_2.

Theorem 4. *Let X be an s.s. on $[0,1]$ such that $X \in \mathbb{K}$ and $X' \in \mathbb{K}$. The following conditions are equivalent*:

(i) *for any sequences $\{f_k\} \subset X$ and $\{f'_k\} \subset X'$ of independent identically distributed functions such that $\int_0^1 f_1(t)\,dt = 0$, the subspaces $[f_k]$ and $[f'_k]$ are complemented in X and X', respectively*;

(ii) $X = L_2[0,1]$.

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Samara State University
e-mail: astashkn@ssu.samara.ru

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