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FUNCTORIAL APPROACH TO INTERPOLATION OF OPERATORS OF WEAK TYPE

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1. If f(t) is a submultiplicative function on the semi-axis $(0, \infty)$, then, as is well known [1, p. 75], there exist the numbers

$$\alpha_f = \lim_{t\to 0+} \frac{\ln f(t)}{\ln t}, \quad \beta_f = \lim_{t\to\infty} \frac{\ln f(t)}{\ln t}.$$

In a particular case, if a function $\varphi(t)$ is positive on $(0, \infty)$, then the function $\mathscr{M}_\varphi(t) = \sup_{s>0} \frac{\varphi(st)}{\varphi(s)}$ is submultiplicative and $\gamma_\varphi = \alpha_{\mathscr{M}_\varphi}$ and $\delta_\varphi = \beta_{\mathscr{M}_\varphi}$ are called respectively the lower and upper dilation exponents of φ. If φ is concave, then $0 \leqslant \gamma_\varphi \leqslant \delta_\varphi \leqslant 1$.

In another approach, let X be a symmetric space of measurable functions on $(0, \infty)$ with Lebesgue measure. Then the dilation operator $\sigma_\tau x(t) = x(\tau^{-1}t)$ is continuous on X and $S(\tau) = \|\sigma_\tau\|_{X\to X}$ is a submultiplicative function on $(0, \infty)$ [1, p. 131]. The numbers $\alpha_X = \alpha_S$ and $\beta_X = \beta_S$ are called respectively Boyd's lower and upper indices of the space X. It is easily verifiable that $0 \le \alpha_X \le \beta_X \le 1$.

Let $\delta(t)$ and $\kappa(t)$ be measurable positive functions on $(0, \infty)$. Given a symmetric space X, we denote by $X_{\kappa,\delta}$ the space of measurable functions x(t) such that $\kappa(t)x^{**}(\omega(t)) \in X$, with the norm

$$\|x\|_{\varkappa,\delta} = \|\varkappa(t)\cdot x^{**}(\delta(t))\|_X \quad \left(x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)\,ds\right).$$

Here $x^*(s)$ is the decreasing rearrangement of $|x(s)|$ [1].

The spaces $X_{\kappa,\delta}$ arise in connection with the fundamental interpolation theorem for symmetric spaces proved by S. G. Krein and E. M. Semenov. Recall that for a positive concave function φ on $(0, \infty)$ the Lorentz space $\Lambda(\varphi)$ is the Banach space of measurement functions on $(0, \infty)$ with the norm

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$$\|x\|_{\Lambda(\varphi)} = \int_0^\infty x^*(t)\,d\varphi(t),$$

and the Marcinkiewicz space is the space with the norm

$$\|x\|_{M(\varphi)} = \sup_{t>0} \frac{\int_0^t x^*(s)\,ds}{\varphi(t)}.$$

Throughout what follows $\tilde{\varphi}(t) = t/\varphi(t)$, $\varphi^{-1}(t) = 1/\varphi(t)$.

Krein–Semenov Theorem [1, Theorem 6.1, p. 175]. Let functions $(0, \infty)$ $\varphi_0(t)$, $\varphi_1(t)$, $\psi_0(t)$ and $\psi_1(t)$ defined on $(0, \infty)$ be positive, concave, and such that

1) the function $\varphi_1(t)/\varphi_0(t)$ is increasing;

2) the range of the function $\psi_0(t)/\psi_1(t)$ contains the domain of the function $\varphi_0(t)/\varphi_1(t)$ $(0 < t < \infty)$.

If a linear operator acts boundedly from $\Lambda(\varphi_0)$ to $M(\tilde{\psi}_0)$ and from $\Lambda(\varphi_1)$ to $M(\tilde{\psi}_1)$, then it acts boundedly from any symmetric space X satisfying the condition $\delta_{\varphi_0} < \alpha_X \leqslant \beta_X < \gamma_{\varphi_1}$ to the space $X_{\varkappa,\delta}$, where $\delta(t)$ is some measurable positive solution of the equation $\psi_0(\delta(t))/\psi_1(\delta(t)) = \varphi_0(t)/\varphi_1(t)$ and $\varkappa(t) = \psi_0(\delta(t))/\varphi_0(t) = \psi_1(\delta(t))/\varphi_1(t)$.

This theorem, closely connected with problems of interpolation of operators of weak type, later has been extended in various ways. We can point out [2], where the couple $(\Lambda(\varphi_0), \Lambda(\varphi_1))$ is replaced by an arbitrary couple of Banach spaces and, which is no less important, it is indicated that interpolation spaces are obtainable by means of the $\mathcal{K}$-method (see [2, Corollary 2]). These topics are very thoroughly covered in [3].

The main goal of the present paper is to clarify the functional meaning of the Krein–Semenov theorem (the definitions of the interpolation spaces and functions see in [1] or [4]). This makes it possible to prove an extension of the theorem in the sense that the Krein–Semenov theorem will apply to the couple (L_1, L_∞) of functions on the semiaxis.

Let E be the Banach ideal space (BIS) [5] of two-sided numerical sequences $a = (a_j)_{j=-\infty}^{\infty}$, and let φ be a nonnegative function on $(0, \infty)$. Then $E(\varphi)$ is a BIS with the norm $\|(a_j)\|_{E(\varphi)} = \|(a_j\varphi(2^j))\|_E$.

Let us briefly recall the definition of the real interpolation method. For a Banach couple (X_0, X_1) and $0 < s, t < \infty$, we define Peetre's $\mathcal{K}$- and $\mathcal{J}$-functionals [6]:

$$\mathcal{K}(s, t, x; X_0, X_1) = \inf_{\substack{x = x_0 + x_1,\\ x_i \in X_i}} \{s\|x_0\|_{X_0} + t\|x_1\|_{X_1}\} \quad (x \in X_0 + X_1);$$

$$\mathcal{J}(s, t, x; X_0, X_1) = \max\{s\|x\|_{X_0}, t\|x\|_{X_1}\} \quad (x \in X_0 \cap X_1).$$

As usual, $\mathcal{K}(t, x; X_0, X_1) = \mathcal{K}(1, t, x; X_0, X_1)$, $\mathcal{J}(t, x; X_0, X_1) = \mathcal{J}(1, t, x; X_0, X_1)$. Let $\ell_\infty \cap \ell_\infty(t^{-1}) \subset E$. The $\mathcal{K}$-method space $(X_0, X_1)_E^{\mathcal{K}}$ consists of all $x \in X_0 + X_1$, for which the norm $\|x\| = \|(\mathcal{K}(2^j, x; X_0, X_1))_j\|_\Sigma$ is finite. If $E \subset \ell_1 + \ell_1(t^{-1})$, then the $\mathcal{J}$-method space $(X_0, X_1)_E^{\mathcal{J}}$ consists of $x \in X_0 + X_1$ admitting the representation

$$x = \sum_{j=-\infty}^{\infty} u_j \quad (\text{convergence in } X_0 + X_1),$$

where $u_j \in X_0 \cap X_1$, and the norm is defined as follows:

$$\|x\| = \inf_{(u_j)} \|(\mathcal{J}(2^j, u_j; X_0, X_1))_j\|_E.$$

A BIS E is called a parameter of the real interpolation method if it is interpolative with respect to the couples $\vec{\ell}_1 = (\ell_1 = (\ell_1, \ell_1(t^{-1}))$ and $\vec{\ell}_\infty = (\ell_\infty, \ell_\infty(t^{-1}))$ [i.e., any linear operator continuous from ℓ_1 to ℓ_∞ and from $\ell_1(t^{-1})$ to $\ell_\infty(t^{-1})$ is continuous in E].

In the sequel, we shall associate with any positive concave function $\varphi(s)$ the function of two variables $\varphi(s, t) = t\varphi(s/t)$ (there will be no confusion from denoting it by the same letter). The function $\varphi(s, t)$ is increasing in each argument and homogeneous, $\varphi(s, 1)=\varphi(s)$, by $\varphi^*(s, t)$ we shall denote the function $1/\varphi(s^{-1}, t^{-1})$. If $E = \ell_1(\tilde{\varphi}^{-1})$, then it is not hard to show that the space $(X_0, X_1)^{\mathscr{J}}_E$ consists of $x \in X_0 + X_1$ expressible in the form $x = \sum_{k=1}^{\infty} u_k$ $(u_k \in X_0 \cap X_1)$, where

$$\sum_{k=1}^{\infty} \varphi(\|u_k\|_{X_0}, \|u_k\|_{X_1}) < \infty,$$

and the norm is equivalent to $\inf \sum_{k=1}^{\infty} \varphi(\|u_k\|_{X_0}, \|u_k\|_{X_1})$ over all such representations.

From the equality $\mathscr{K}(t, x; L_1, L_\infty) = \int_0^t x^*(s)\,ds$ (see, e.g., [1, p. 108]) it follows that for any positive concave function φ we have

$$(L_1, L_\infty)^{\mathscr{K}}_{l_\infty(\varphi^{-1})} = \mathrm{M}(\varphi).$$

Analogously

$$(L_1, L_\infty)^{\mathscr{J}}_{l_1(\tilde{\varphi}^{-1})} = \overset{0}{\Lambda}(\varphi),$$

where $\overset{0}{\Lambda}(\varphi)$ is the closure of $L_1 \cap L_\infty$ in $\Lambda(\varphi)$ with the induced norm. The equality $\overset{0}{\Lambda}(\varphi)=\Lambda(\varphi)$ holds if and only if $\lim_{t\to+\infty} \varphi(t) = \infty$.

2. Let us first show how the spaces interpolative between the couples $(\Lambda(\varphi_0), \Lambda(\varphi_1))$ and $(\mathrm{M}(\tilde{\varphi}_0), \mathrm{M}(\tilde{\varphi}_1))$ can be described with the aid of the real interpolation method.

THEOREM 1. Let φ_0 and φ_1 be positive concave functions such that φ_1/φ_0 is increasing and $\delta_{\varphi_0} < \gamma_{\varphi_1}$. If Boyd's indices of a symmetric space X satisfy the conditions

$$\delta_{\varphi_0} < \alpha_X \leqslant \beta_X < \gamma_{\varphi_1},$$

then X can be constructed by the $\mathscr{K}$-method from the corresponding couples of Lorentz and Marcinkiewicz spaces:

$$X = (\overset{0}{\Lambda}(\varphi_0), \overset{0}{\Lambda}(\varphi_1))^{\mathscr{K}}_E = (\mathrm{M}(\tilde{\varphi}_0), \mathrm{M}(\tilde{\varphi}_1))^{\mathscr{K}}_E,$$

where E is a parameter of the real method.

Proof. First consider this particular case: $X = M_{1-\theta} = \mathrm{M}(\tilde{\varphi})$, where $\varphi(t) = t^\theta$, $\delta_{\varphi_0} < \theta < \gamma_{\varphi_1}$. Here the space X coincides with the space $L_{p,\infty}$ for $p = 1/\theta$ [1]. If we put

$$\rho(u, v) = \inf_{s>0} s^\theta \left(\frac{u}{\varphi_0(s)} + \frac{v}{\varphi_1(s)}\right),$$

then $t^\theta = \rho(\varphi_0(t), \varphi_1(t))$. A straightforward computation shows that dilation exponents of the function ρ are nontrivial, i.e., $0 < (\theta - \delta_{\varphi_0})/(1 - \delta_{\varphi_0}) \leqslant \gamma_\rho \leqslant \delta_\rho \leqslant \theta/\gamma_{\varphi_1} < 1$. Therefore, from the results of [7] it follows that

$$\mathrm{M}_{1-\theta} = (\overset{0}{\Lambda}(\varphi_0), \overset{0}{\Lambda}(\varphi_1))^{\mathscr{K}}_{l_\infty(\tilde{\rho}^{-1})},$$

$$\mathrm{M}_{1-\theta} = (\mathrm{M}(\tilde{\varphi}_0), \mathrm{M}(\tilde{\varphi}_1))^{\mathscr{K}}_{l_\infty(\tilde{\rho}^{-1})},$$

i.e., in this case the space $\ell_\infty(\tilde{\rho}^{-1})$ is a common parameter E.

We now turn to the general case. Since $\delta_{\varphi_0} < \alpha_X \leqslant \beta_X < \gamma_{\varphi_1}$ there can be found θ_0 and θ_1 such that $\delta_{\varphi_0} < \theta_0 < \alpha_X \leqslant \beta_X < \theta_1 < \gamma_{\varphi_1}$. From the inequalities $\theta_0 < \alpha_X \leq \beta_X < \theta_1$ and the Krein–Semenov theorem it follows that the space X is interpolative between $M_{1-\theta_0}$ and $M_{1-\theta_1}$, consequently $X = \mathscr{F}(M_{1-\theta_0}, M_{1-\theta_1})$, where $\mathscr{F}$ is some interpolation functor (see, e.g., [1]). Hence

$$X = \mathscr{F}\left((\overset{0}{\Lambda}(\varphi_0), \overset{0}{\Lambda}(\varphi_1))^{\mathscr{K}}_{l_\infty(\tilde{\rho}_0^{-1})}, (\overset{0}{\Lambda}(\varphi_0)\, \overset{0}{\Lambda}(\varphi_1))^{\mathscr{K}}_{l_\infty(\tilde{\rho}_1^{-1})}\right) = \tag{1}$$

$$= \mathscr{F}\left((\mathrm{M}(\tilde{\varphi}_0), \mathrm{M}(\tilde{\varphi}_1))^{\mathscr{K}}_{l_\infty(\tilde{\rho}_0^{-1})}, (\mathrm{M}(\tilde{\varphi}_0), \mathrm{M}(\tilde{\varphi}_1))^{\mathscr{K}}_{l_\infty(\tilde{\rho}_1^{-1})}\right), \tag{1}$$

where $t^{\theta_0} = \rho_0(\varphi_0(t), \varphi_1(t))$, $t^{\theta_1} = \rho_1(\varphi_0(t), \varphi_1(t))$. Since the dilation exponents of the functions ρ_0 and ρ_1 are nontrivial, $\ell_\infty(\tilde{\rho}_0^{-1})$ and $\ell_\infty(\tilde{\rho}_1^{-1})$ are spaces-parameters of the real method [8], therefore the reiteration theorem for an arbitrary functor $\mathscr{F}$ [9] applies to (1), which yields

$$X = (\overset{0}{\Lambda}(\varphi_0), \overset{0}{\Lambda}(\varphi_1))^{\mathscr{K}}_{\mathscr{F}(l_\infty(\tilde{\rho}_0^{-1}), l_\infty(\tilde{\rho}_1^{-1}))},$$
$$X = (\mathrm{M}(\tilde{\varphi}_0), \mathrm{M}(\tilde{\varphi}_1))^{\mathscr{K}}_{\mathscr{F}(l_\infty(\tilde{\rho}_0^{-1}), l_\infty(\tilde{\rho}_1^{-1}))}.$$

Thus the space $\mathscr{F}(\ell_\infty(\tilde{\rho}_0^{-1}), \ell_\infty(\tilde{\rho}_1^{-1}))$ can be taken as a common parameter E. The theorem is proved.

From the results of [10] it is not hard to deduce that the space-parameter E in Theorem 1 is uniquely determined. Furthermore, there holds

Proposition. The space E consists of all the sequences $a = (a_k)_{k=-\infty}^{\infty}$ such that $\mathscr{K}(\varphi_0^{-1}(t), \varphi_1^{-1}(t), a; \vec{l}_\infty) \in X$.

Proof. Denote by G the space of sequences $a = (a_k)$ for which $\mathscr{K}(\varphi_0^{-1}(t), \varphi_1^{-1}(t), a; \vec{l}_\infty) \in X$ and show first that $E \subset G$.

On the space $\ell_\infty + \ell_\infty(t^{-1})$ consider the sublinear operator P that carries a sequence $a = (a_k)_{k=-\infty}^{\infty}$ into the function $\mathscr{K}(\varphi_0^{-1}(t), \varphi_1^{-1}(t), a; \vec{l}_\infty$ on the semiaxis $(0, \infty)$. We shall show that the operator P acts from $\ell_\infty(\tilde{\rho}_0^{-1})$ to $M_{1-\theta_0}$ and from $\ell_\infty(\tilde{\rho}_1^{-1})$ to $M_{1-\theta_1}$, where θ_0, θ_1 and the functions ρ_0, ρ_1 are like those in the proof of Theorem 1.

Indeed,

$$\mathscr{K}(u, v, (a_k); \vec{l}_\infty) \leqslant 2 \sup_{k \in Z} \min(u, 2^{-k}v) |a_k| \leqslant 2 \|a\|_{l_\infty(\tilde{\rho}_i^{-1})} \sup_k \frac{\min(u, 2^{-k}v)}{\rho_i(1, 2^{-k})} \leqslant 2\rho_i^*(u, v) \|a\|_{l_\infty(\tilde{\rho}_i^{-1})}$$

for i = 0, 1. Therefore

$$\mathscr{K}(\varphi_0^{-1}(t), \varphi_1^{-1}(t), a; \vec{l}_\infty) \leqslant \frac{2}{\rho_i(\varphi_0(t), \varphi_1(t))} \|a\|_{l_\infty(\tilde{\rho}_i^{-1})} = \frac{2}{t^{\theta_i}} \|a\|_{l_\infty(\tilde{\rho}_i^{-1})}.$$

Consequently,

$$\|Pa\|_{M_{1-\theta_i}} \leqslant 2\|a\|_{l_\infty(\tilde{\rho}_i^{-1})} \|t^{-\theta_i}\|_{M_{1-\theta_i}} = \frac{2}{1-\theta_i} \|a\|_{l_\infty(\tilde{\rho}_i^{-1})}.$$

Let now $a' = (a_k') \in E$. For the function $P(a')(t)$ find a linear operator T: $\ell_\infty + \ell_\infty(t^{-1}) \to S(0, \infty)$ [the space of functions on $(0, \infty)$ measurable and almost everywhere finite] such that $T(a') = P(a')$ and $|T(a)| \leqslant P(a)$ $(a \in l_\infty + l_\infty(t^{-1}))$ (this extension of the Hahn–Banach theorem to functions with values in $\mathscr{K}$-spaces is due to Kantorovich (see [11])). Like P, the operator T acts from $\ell_\infty(\tilde{\rho}_i^{-1})$ to $M_{1-\theta_i}$ (i = 0, 1), therefore also from the space $E = \mathscr{F}(\ell_\infty(\tilde{\rho}_0^{-1}), \ell_\infty(\tilde{\rho}_1^{-1}))$ to $\mathscr{F}(M_{1-\theta_0}, M_{1-\theta_1}) = X$, since $\mathscr{F}$ is an interpolation functor. Thus, if $a' \in E$, then $P(a') \in X$, i.e., $a' \in G$.

Prove the reverse inclusion: $G \subset E$. By Theorem 1, $X = (\mathrm{M}(\tilde{\varphi}_0), \mathrm{M}(\tilde{\varphi}_1))^{\mathscr{K}}_E$, i.e., if $a \in G$, then

$$f(t) = \mathscr{K}(\varphi_0^{-1}(t), \varphi_1^{-1}(t), a; \vec{l}_\infty) \in (\mathrm{M}(\tilde{\varphi}_0), \mathrm{M}(\tilde{\varphi}_1))^{\mathscr{K}}_E$$

or

$$(\mathscr{K}(2^n, f; \{\mathrm{M}(\tilde{\varphi}_0), \mathrm{M}(\tilde{\varphi}_1)\}))_n \in E. \tag{2}$$

As is well known (see, e.g., [1]), a $\mathscr{K}$-functional in a couple of Marcinkiewicz spaces is computable by the formula

$$\mathscr{K}(u, v, f; \{\mathrm{M}(\tilde{\varphi}_0), \mathrm{M}(\tilde{\varphi}_1)\}) \asymp \sup_{t>0} \frac{\int_0^t f^*(s)\, ds}{\max(\tilde{\varphi}_0(t)/u, \tilde{\varphi}_1(t)/v)} = \sup_{t>0} f^{**}(t) \min(u\varphi_0(t), v\varphi_1(t)).$$

Since $f^{**}(t) \geq f^*(t) = f(t)$, we have

$$\mathscr{K}(u, v, f; \{M(\tilde{\varphi}_0), M(\tilde{\varphi}_1)\}) \geqslant \sup_{t>0} f(t) \min(u\varphi_0(t), v\varphi_1(t)) =$$

$$= \sup_{t>0} \mathscr{K}(\varphi_0^{-1}(t), \varphi_1^{-1}(t), a; \vec{l}_\infty) \min(u\varphi_0(t), v\varphi_1(t)) = \mathscr{K}(u, v, a; \vec{l}_\infty).$$

The last equality follows from the fact that due to the condition $\delta_{\varphi_0} < \gamma_{\varphi_1}$ the range of the ratio φ_1/φ_0 is the whole semiaxis $(0, \infty)$.

Thus, it follows from (2) that $(\mathscr{K}(2^n, a; \vec{l}_\infty)) \in E$ and since $\mathscr{K}(2^n, a; \vec{l}_\infty) \geqslant |a_n|$ $a \in E$. The proposition is proved.

Now let us show how Theorem 1 makes it possible to obtain a functorial interpretation of the Krein–Semenov theorem. For this, it suffices to investigate the space

$$(M(\tilde{\psi}_0), M(\tilde{\psi}_1))_E^{\mathscr{K}},$$

where

$$(\overset{0}{\Lambda}(\varphi_0), \overset{0}{\Lambda}(\varphi_1))_E^{\mathscr{K}} = (M(\tilde{\varphi}_0), M(\tilde{\varphi}_1))_E^{\mathscr{K}} = X. \tag{3}$$

If $y \in (M(\tilde{\psi}_0), M(\tilde{\psi}))_E^{\mathscr{K}}$, then $(a_n) = (\mathscr{K}(2^n, y; \{M(\tilde{\psi}_0), M(\tilde{\psi}_1)\})) \in E$ hence by the proposition (see above) we have

$$\mathscr{K}(\varphi_0^{-1}(t), \varphi_1^{-1}(t), (a_n); \vec{l}_\infty) \in X,$$

i.e.,

$$\sup_{n \in \mathbf{Z}} a_n \min(\varphi_0^{-1}(t), 2^{-n}\varphi_1^{-1}(t)) = \sup_n \mathscr{K}(2^n, y; \{M(\tilde{\psi}_0), M(\tilde{\psi}_1)\}) \min(\varphi_0^{-1}(t), 2^{-n}\varphi_1^{-1}(t)) \in X.$$

Since for any quasiconcave function φ we have

$$\varphi(u, v) \asymp \sup_{n \in \mathbf{Z}} \varphi(1, 2^n) \min(u, 2^{-n}v),$$

the foregoing implies the inclusion $\mathscr{K}(\varphi_0^{-1}(t), \varphi_1^{-1}(t), y; \{M(\tilde{\psi}_0), M(\tilde{\psi}_1)\}) \in X$. Then by formula of a $\mathscr{K}$-functional for a couple of Marcinkiewicz spaces we have

$$\sup_{u>0} \frac{\frac{1}{u}\int_0^u y^*(s)\,ds}{\max\{\varphi_0(t)/\psi_0(t), \varphi_1(t)/\psi_1(t)\}} \in X.$$

If the range of the ratio ψ_1/ψ_0 coincides with the range of φ_1/φ_0 then for any t there will be found u_t such that $\varphi_0(t)/\psi_0(u_t) = \varphi_1(t)/\psi_1(u_t)$. Consequently, if $y \in (M(\tilde{\psi}_0), M(\tilde{\psi}_1))_E^{\mathscr{K}}$, then $(\psi_0(u_t)/\varphi_0(t))y^{**}(u_t) \in X$.

Turning to the notation of the Krein–Semenov theorem, we finally obtain

$$(M(\tilde{\psi}_0), M(\tilde{\psi}_1))_E^{\mathscr{K}} \subset X_{\varkappa,\delta}, \tag{4}$$

where $\delta(t) = u_t$, and $\kappa(t) = \psi_0(u_t)/\varphi_0(t)$. Since $\overset{0}{\Lambda}(\varphi) \subset \Lambda(\varphi) \subset M(\tilde{\varphi})$ from (3) we have $(\Lambda(\varphi_0), \Lambda(\varphi_1))_E^{\mathscr{K}} = X$. Thus, the Krein–Semenov theorem is obtainable as the result of the application of a functor $(\cdot, \cdot)_E^{\mathscr{K}}$ to the couples $(\Lambda(\varphi_0), \Lambda(\varphi_1))$ and $(M(\tilde{\psi}_0), M(\tilde{\psi}_1))$, i.e., if a linear operator T acts from $\Lambda(\varphi_i)$ to $M(\tilde{\psi}_i)$ $(i = 0, 1)$, then

$$T\colon X = (\Lambda(\varphi_0), \Lambda(\varphi_1))_E^{\mathscr{K}} \to (M(\tilde{\psi}_0), M(\tilde{\psi}_1))_E^{\mathscr{K}} \subset X_{\varkappa,\delta},$$

where E is a suitable parameter of the real interpolation method.

3. The discrete spaces-parameters of the $\mathscr{K}$-method are not very convenient in studying the real interpolation method, since the spaces interpolative between the weighted spaces ℓ_∞ and $\ell_\infty(t^{-1})$ are as yet imperfectly understood. It turns out that the symmetric spaces interpolative between L_1 and L_∞, whose theory if well developed (see, e.g., [1]), can serve as natural parameters for the $\mathscr{K}$-method.

Let X be a symmetric interpolative between L_1 and L_∞ space of measurable functions on $(0, \infty)$. Following [12], we define the space $X(\vec{A})$, where $\vec{A} = (A_0, A_1)$ is an arbitrary Banach couple:

$$X(\vec{A}) = \left\{a \in A_0 + A_1 \colon \mathscr{K}(t, a; \vec{A}) \xrightarrow[t \to 0]{} 0 \text{ и } \frac{d}{dt}\mathscr{K}(t, a; \vec{A}) \in X\right\}$$

with the norm $\|a\|_{X(\vec{A})} = \left\|\frac{d}{dt}\mathscr{K}(t, a; \vec{A})\right\|_X$. Specifically, $X(L_1, L_\infty) = X$. Since interpolation in the couple (L_1, L_∞) is describable by the $\mathscr{K}$-method, there will be found a space F [13] interpolative between ℓ_∞ and $\ell_\infty(t^{-1})$ and such that $X = (L_1, L_\infty)_F^{\mathscr{K}}$. Therefore, $\|x\|_X = \left\|\left(\int_0^{2^k} x^*(s)\,ds\right)_k\right\|_F$ and $a \in X(\vec{A})$ mean that $(\mathscr{K}(2^k, a; \vec{A})) \in F \cap (c_0 + l_\infty(t^{-1}))$ and $\|a\|_{X(\vec{A})} = \|(\mathscr{K}(2^k, a; \vec{A}))\|_F$.

Thus

$$X(\vec{A}) = (A_0, A_1)^{\mathscr{K}}_{F \cap (c_0 + l_\infty(t^{-1}))}.$$

The last equality shows, among other things, that in what follows we can assume that $F \subset c_0 + \ell_\infty(t^{-1})$.

Thus, any space of the form $X(\vec{A})$ is a $\mathscr{K}$-method space. It is not hard to see that the converse is partly valid: For $F \subset c_0 + \ell_\infty(t^{-1})$

$$(A_0, A_1)_F^{\mathscr{K}} = X(\vec{A}), \tag{5}$$

where

$$X = (L_1, L_\infty)_F^{\mathscr{K}}. \tag{6}$$

We shall show that an extension of the Krein–Semenov theorem holds for the spaces $X(\vec{A})$. The proof is based on the functorial interpretation of that theorem obtained in Sec. 2.

<u>THEOREM 2</u>. Let $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ be arbitrary Banach couples, let $\varphi_0, \varphi_1, \psi_0, \psi_1$ be positive concave functions like those in the Krein–Semenov theorem, and let X be a symmetric space whose Boyd indices satisfy the relations $\delta_{\varphi_0} < \alpha_X \leqslant \beta_X < \gamma_{\varphi_1}$. Then any linear operator bounded from $\overset{0}{\Lambda}(\varphi_i)(\vec{A})$ to $M(\tilde{\psi}_i)(\vec{B})$ $(i = 0, 1)$, will boundedly act from the space $X(\vec{A})$ to $X_{\kappa,\delta}(\vec{B})$, where κ and δ are constructed from $\varphi_0, \varphi_1, \psi_0, \psi_1$ in the same way as in the Krein–Semenov theorem.

<u>Proof</u>. Using Theorem 1, for space X find a parameter E of the real method such that $(\overset{0}{\Lambda}(\varphi_0), \overset{0}{\Lambda}(\varphi_1))_E^{\mathscr{K}} = X$. As was observed, any space of the form $X(\vec{A})$ is constructible by the $\mathscr{K}$-method, specifically

$$\overset{0}{\Lambda}(\varphi_0)(\vec{A}) = \vec{A}^{\mathscr{K}}_{F_0}, \quad \overset{0}{\Lambda}(\varphi_1)(\vec{A}) = \vec{A}^{\mathscr{K}}_{F_1}.$$

By the reiteration theorem for the $\mathscr{K}$-method [13],

$$(\overset{0}{\Lambda}(\varphi_0)(\vec{A}), \overset{0}{\Lambda}(\varphi_1)(\vec{A}))_E^{\mathscr{K}} = \left(\vec{A}^{\mathscr{K}}_{F_0}, \vec{A}^{\mathscr{K}}_{F_1}\right)_E^{\mathscr{K}} = \vec{A}^{\mathscr{K}}_{(F_0, F_1)_E^{\mathscr{K}}}.$$

In the case $\vec{A} = (L_1, L_\infty)$ we obtain

$$X = (\overset{0}{\Lambda}(\varphi_0), \overset{0}{\Lambda}(\varphi_1))_E^{\mathscr{K}} = (L_1, L_\infty)^{\mathscr{K}}_{(F_0, F_1)_E^{\mathscr{K}}}.$$

Thus, from formulas (5), (6) we have

$$(\overset{0}{\Lambda}(\varphi_0)(\vec{A}), \overset{0}{\Lambda}(\varphi_1)(\vec{A}))_E^{\mathscr{K}} = \vec{A}^{\mathscr{K}}_{(F_0, F_1)_E^{\mathscr{K}}} = X(\vec{A}). \tag{7}$$

Analogously

$$(M(\tilde{\psi}_0)(\vec{B}), M(\tilde{\psi}_1)(\vec{B}))_E^{\mathscr{K}} = (M(\tilde{\psi}_0), M(\tilde{\psi}_1))_E^{\mathscr{K}}(\vec{B}),$$

therefore, in view of (4),

$$(\mathrm{M}(\tilde{\psi}_0)(\vec{B}), \mathrm{M}(\tilde{\psi}_1)(\vec{B}))_E^{\mathscr{K}} \subset X_{\varkappa,\delta}(\vec{B}). \tag{8}$$

The assertion of Theorem 2 immediately follows from relations (7) and (8).

Theorem 2 can be somewhat strengthened. Using the relation between the $\mathscr{K}$- and $\mathscr{J}$-methods of interpolation (see [13]), it is not hard to show that the equality

$$\overset{0}{\Lambda}(\varphi)(A_0, A_1) = (\tilde{A}_0, \tilde{A}_1)^{\mathscr{J}}_{l_1(\tilde{\varphi}^{-1})},$$

where $\tilde{A}_i$ is the completion of A_i with respect to the sum $A_0 + A_1$, holds for any positive concave function φ and any Banach couple $\vec{A} = (A_0, A_1)$. Therefore

$$(A_0, A_1)^{\mathscr{J}}_{l_1(\tilde{\varphi}^{-1})} \subset \overset{0}{\Lambda}(\varphi)(A_0, A_1).$$

On the other hand, it is not hard to see that

$$\mathrm{M}(\psi)(\vec{B}) \subset \vec{B}^{\mathscr{K}}_{l_\infty(\psi^{-1})}$$

for any positive concave function ψ.

THEOREM 3. Let functions $\varphi_0, \varphi_1, \psi_0, \psi_1$ and a space X satisfy the hypotheses of the Krein–Semenov theorem, and let $\vec{A} = (A_0, A_1)$, $\vec{B} = (B_0, B_1)$ be Banach couples. Then any linear operator bounded from $(A_0, A_1)^{\mathscr{J}}_{l_1(\tilde{\varphi}_i^{-1})}$ to $(B_0, B_1)^{\mathscr{K}}_{l_\infty(\tilde{\psi}_i^{-1})}$ will boundedly act from $X(\vec{A})$ to $X_{\kappa,\delta}(\vec{B})$.

Proof. In analogy with the proof of Theorem 2, consider the space

$$Y = \left(\vec{A}^{\mathscr{J}}_{l_1(\tilde{\varphi}_0^{-1})}, \vec{A}^{\mathscr{J}}_{l_1(\tilde{\varphi}_1^{-1})}\right)^{\mathscr{K}}_E.$$

Since E is a parameter of the real method, by the reiteration theorem in [14] we have

$$\left(\vec{A}^{\mathscr{J}}_{l_1(\tilde{\varphi}_0^{-1})}, \vec{A}^{\mathscr{J}}_{l_1(\tilde{\varphi}_1^{-1})}\right)^{\mathscr{K}}_E = \vec{A}^{\mathscr{J}}_{(l_1(\tilde{\varphi}_0^{-1}), l_1(\tilde{\varphi}_1^{-1}))^{\mathscr{K}}_E} = \vec{A}^{\mathscr{K}}_{(l_1(\tilde{\varphi}_0^{-1}), l_1(\tilde{\varphi}_1^{-1}))^{\mathscr{K}}_E}.$$

Specifically, for the couple $\vec{A} = (L_1, L_\infty)$,

$$X = (\overset{0}{\Lambda}(\varphi_0), \overset{0}{\Lambda}(\varphi_1))^{\mathscr{K}}_E = (L_1, L_\infty)^{\mathscr{K}}_{(l_1(\tilde{\varphi}_0^{-1}), l_1(\tilde{\varphi}_1^{-1}))^{\mathscr{K}}_E},$$

whence, as before, $Y = X(\vec{A})$. Analogously,

$$\left(\vec{B}^{\mathscr{K}}_{l_\infty(\tilde{\varphi}_0^{-1})}, \vec{B}^{\mathscr{K}}_{l_\infty(\tilde{\varphi}_1^{-1})}\right)^{\mathscr{K}}_E \subset X_{\varkappa,\delta}(\vec{B}).$$

The theorem is proved.

4. In conclusion we shall show how with the aid of the above results the theorem on the continuity of bilinear functionals can be proved.

Given a symmetric space X on $(0, \infty)$, we denote by X' the space associated with X, i.e.,

$$\|y\|_{X'} := \sup_{\|x\|_X \leqslant 1} \int_0^\infty y(t)\, x(t)\, dt.$$

THEOREM 4. Let $\varphi_0, \varphi_1, \psi_0, \psi_1$ satisfy the hypotheses of the Krein–Semenov theorem, let $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ be Banach couples, and let $B_0 \cap B_1$ be everywhere dense in B_0 and B_1. Let, further, $R = R(a, b)$ be a bilinear functional defined on $(A_0 \cap A_1) \times (B_0 \cap B_1)$, and such that

$$|R(a, b)| \leqslant C \min_{i=0,1} \varphi_i(\|a\|_{A_0}, \|a\|_{A_1})\, \psi_i^*(\|b\|_{B_0}, \|b\|_{B_1}).$$

for $a \in A_0 \cap A_1$ and $b \in B_0 \cap B_1$. If $Y = X'$ is a symmetric space on $(0, \infty)$ whose Boyd indices satisfy

$$\delta_{\varphi_0} < \alpha_Y \leqslant \beta_Y < \gamma_{\varphi_1},$$

then R can be extended to $X'(A_0, A_1) \times X_{\kappa_1,\delta}(B_1, B_0)$ so that

$$|R(a, b)| \leqslant C \| a \|_{X'(\vec{A})} \| b \|_{X_{\varkappa_1,\delta}(B_1,B_0)},$$

where $\kappa_1(t) = \delta'(t)/\kappa(t)$ (κ and δ are determined by φ_0, φ_1, ψ_0 and ψ_1 in the same way as in the Krein–Semenov theorem, and $\delta(t)$ is assumed monotonous and absolutely continuous).

First let us prove two lemmas.

Denote by E' the space associated with the BIS of sequences E, i.e., the set of all sequences (b_k) for which

$$\|(b_k)\|_{E'} = \sup_{\|(a_k)\|_E \leqslant 1} \sum_{k=-\infty}^{\infty} a_k b_k < \infty.$$

<u>LEMMA 1</u>. If E is interpolative between ℓ_∞ and $\ell_\infty(t^{-1})$, $\ell_\infty \notin E$, then

$$\|(b_k)\|_{E'} \asymp \sup_{\|x\|_X \leqslant 1} \sum_{k=-\infty}^{\infty} b_k \int_0^{2^k} x^*(t)\,dt,$$

where $X = (L_1, L_\infty)_E^{\mathscr{K}}$.

<u>Proof</u>. Since

$$\|x\|_X = \left\| \left(\int_0^{2^k} x^*(t)\,dt \right)_k \right\|_E,$$

we have

$$\sup_{\|x\|_X \leqslant 1} \sum_{k=-\infty}^{\infty} b_k \int_0^{2^k} x^*(t)\,dt \leqslant \|(b_k)\|_{E'}.$$

On the other hand, since E is interpolative between ℓ_∞ and $\ell_\infty(t^{-1})$, there can be found $C_1 > 0$ such that for any $a \in E$ the element $a' = (a_k')$,

$$a_k' = \sup_{j \in Z} |a_j| \min(1, 2^{k-j})$$

will also belong to E and $\|a'\|_E \leqslant C_1 \|a\|_E$ (see [13, Sec. 4]). Furthermore, $|a_k| \leqslant a_k'$ and, in view of the monotonicity of (a_k') and the condition $\ell_\infty \notin E$,

$$\lim_{k \to -\infty} a_k' = 0. \tag{9}$$

The function

$$f_a(t) = \sum_{k=-\infty}^{\infty} a_k' \chi_{[2^k, 2^{k+1})}(t) \quad \left(\chi_e(t) = \begin{cases} 1, & t \in e, \\ 0, & t \notin e, \end{cases} \right)$$

is increasing, and if $t_1 < t_2$, then

$$f_a(t_2)/t_2 \leqslant 2 f_a(t_1)/t_1.$$

Consequently, f_a is equivalent to its minimal concave majorant $\bar{f}_a$ [1, p. 63]. From (9) it follows that $\bar{f}_a$ is absolutely continuous on $[0, \infty)$, therefore for some $C_2 > 0$

$$\int_0^{2^k} (\bar{f}_a)'(t)\,dt = f_a(2^k) \leqslant C_2 f(2^k) = C_2 a_k'.$$

Hence

$$\|(f_a)'\|_X \leqslant C_2\|(a_k')\|_E \leqslant C_1C_2\|(a_k)\|_E,$$

$$\left|\sum_{k=-\infty}^{\infty} a_k b_k\right| \leqslant \sum_{k=-\infty}^{\infty} |b_k|\, a_k' \leqslant \sum_{k=-\infty}^{\infty} |b_k| \int_0^{2^k} (f_a)'(t)\,dt.$$

The lemma is proved.

Let X be a symmetric space on $(0, \infty)$. We denote by X^+ the set of all functions $y(t)$ measurable on $(0, \infty)$ and such that

$$\|y\|_{X^+} = \sup_{\|x\|_X \leqslant 1} \int_0^\infty y^{**}(t)\, x^{**}(t)\,dt < \infty;$$

X^+ is a symmetric space; if $0 < \alpha_X \leq \beta_X < 1$, then X^+ coincides with X'.

Given a Banach couple $\vec{A} = (A_0, A_1)$, we shall denote by $\vec{A}^t$ the couple (A_1, A_0).

LEMMA 2. Let $\vec{A} = (A_0, A_1)$ be a Banach couple, and let $A_0 \cap A_1$ be everywhere dense in A_0 and A_1. If X is interpolative between L_1 and L_∞, then

$$(X(\vec{A}))^* \supset X^+(\vec{A}^{*t}),$$

where $\vec{A}^* = (A_0^*, A_1^*)$.

Proof. As has been indicated, we can assume that $X = (L_1, L_\infty)_F^{\mathscr{K}}$, where F is interpolative between ℓ_∞ and $\ell_\infty(t^{-1})$ and $F \subset c_0 + \ell_\infty(t^{-1})$. Let $a \in A_0 + A_1$, $a = a_0^k + a_1^k$, $a_i^k \in A_i$ $(k \in \mathbf{Z})$, $a^* = \sum_{k=-\infty}^{\infty} a_k^*$, $a_k^* \in A_0^* \cap A_1^*$ (convergence in $A_0^* + A_1^*$). Then

$$|a^*(a)| \leqslant \sum_{k=-\infty}^{\infty} \left(|a_k^*(a_0^k)| + |a_k^*(a_1^k)|\right) \leqslant \sum_{k=-\infty}^{\infty} \max\left(\|a_k^*\|_{A_0^*}, 2^{-k}\|a_k^*\|_{A_1^*}\right)\left(\|a_0^k\|_{A_0} + 2^k\|a_1^k\|_{A_1}\right),$$

whence, passing to the infimum in each summand, we obtain

$$|a^*(a)| \leqslant \sum_{k=-\infty}^{\infty} \mathscr{J}(2^{-k}, a_k^*; \vec{A}^*)\, \mathscr{K}(2^k, a; \vec{A}).$$

From $\|a\|_{X(\vec{A})} = \|(\mathscr{K}(2^j, a; \vec{A}))_j\|_F$ and the definition of the $\mathscr{J}$-method we have

$$|a^*(a)| \leqslant \|a^*\|_{(A_0^*, A_1^*)_{F_+}^{\mathscr{J}}} \|a\|_{X(\vec{A})}$$

or

$$(X(\vec{A}))^* \supset (A_0^*, A_1^*)_{F_+}^{\mathscr{J}}, \tag{10}$$

where $F_+ = \{(a_k) : (a_{-k}) \in F'\}$ with the norm $\|(a_k)\|_{F_+} = \|(a_{-k})\|_{F'}$.

The BIS F is interpolative between ℓ_∞ and $\ell_\infty(t^{-1})$. Consequently, $\ell_\infty \cap \ell_\infty(t^{-1}) \subset F$, $F' \subset \ell_1 + \ell_1(t)$ and $F_+ \subset \ell_1 + \ell_1(t^{-1})$. Therefore, $a^* \in (A_0^*, A_1^*)_{F_+}^{\mathscr{K}}$ satisfy the hypotheses of the fundamental lemma of the interpolation theory [4]

$$\lim_{t \to 0+} \mathscr{K}(1, t, a^*, \vec{A}^*) = \lim_{t \to 0+} \mathscr{K}(t, 1, a^*; \vec{A}^*) = 0,$$

by which theory we have

$$(A_0^*, A_1^*)_{F_+}^{\mathscr{K}} \subset (A_0^*, A_1^*)_{F_+}^{\mathscr{J}}. \tag{11}$$

Since $\mathscr{K}(t, b; \vec{B}) = t\mathscr{K}(1/t, b; \vec{B}^t)$ it is easy to see that

$$(A_0^*, A_1^*)_{F_+}^{\mathscr{K}} = (A_1^*, A_0^*)_{F'(t^{-1})}^{\mathscr{K}}. \tag{12}$$

From (10)-(12) it follows that

$$(X(\vec{A}))^* \supset Y(\vec{A}^{*t}),$$

where $Y=(L_1, L_\infty)^{\mathscr{K}}_{F'(t^{-1})}$.

It remains to show that $Y = X^+$. Indeed, by Lemma 1

$$\|y\|_Y=\left\|\left(\int\limits_0^{2^k} y^*(s)\,ds\right)_k\right\|_{F'(t^{-1})}=\sup_{\|(a_k)\|_F\leqslant 1}\sum_{k=-\infty}^{\infty}2^{-k}\int\limits_0^{2^k} y^*(s)\,ds a_k \asymp \sup_{\|x\|_X\leqslant 1}\sum_{k=-\infty}^{\infty}2^{-k}\int\limits_0^{2^k} y^*(s)\,ds\int\limits_0^{2^k} x^*(s)\,ds=$$

$$=\sup_{\|x\|_X\leqslant 1}\sum_{k=-\infty}^{\infty}2^k y^{**}(2^k)\,x^{**}(2^k)\asymp \sup_{\|x\|_X\leqslant 1}\int\limits_0^{\infty} x^{**}(t)\,y^{**}(t)\,dt=\|y\|_{X^+}.$$

The lemma is proved.

Proof of Theorem 4. First verify that R can be extended to a continuous functional on

$$(A_0, A_1)^{\mathscr{J}}_{l_1(\tilde{\varphi}_i^{-1})}\times (B_0, B_1)^{\mathscr{J}}_{l_1(\tilde{\psi}_i(t^{-1}))}\quad (i=0,1).$$

Indeed, if $a=\sum\limits_{k=1}^{\infty}a_k$ $(a_k\in A_0\cap A_1)$ and $b=\sum\limits_{j=1}^{\infty}b_j$ $(b_j\in B_0\cap B_1)$ then

$$|R(a,b)|\leqslant \sum_{k,j=1}^{\infty}|R(a_k,b_j)|\leqslant C\sum_{k=1}^{\infty}\varphi_i\left(\|a_k\|_{A_0},\|a_k\|_{A_1}\right)\sum_{j=1}^{\infty}\psi_i^*\left(\|b_j\|_{B_0},\|b_j\|_{B_1}\right).$$

It remains to take the infimum of all possible representations of a and b and make use of the definition of the $\mathscr{J}$-method spaces for the case of ℓ_1-parameters (see Sec. 1).

Thus

$$|R(a,b)|\leqslant C\|a\|_{U_i}\|b\|_{V_i}\quad (i=0,1),$$

where $U_i=(A_0,A_1)^{\mathscr{J}}_{l_1(\tilde{\varphi}_i^{-1})}$, $V_i=(B_0,B_1)^{\mathscr{J}}_{l_1(\tilde{\psi}_i(t^{-1}))}$. Consequently, for a fixed $a\in U_i$ $R(a,\cdot)\in V_i^*$ and for the corresponding linear operator T: $U_i \to V_i^*$ we have

$$\|Ta\|_{V_i^*}\leqslant C\|a\|_{U_i}\quad (i=0,1).$$

Due to the duality between the $\mathscr{J}$- and $\mathscr{K}$-methods of interpolation (see [12]), we have

$$V_i^*=\left(B_0^*,B_1^*\right)^{\mathscr{K}}_{l_\infty(\tilde{\psi}_i^{-1})},$$

therefore, by Theorem 3,

$$\|Ta\|_{X'_{\varkappa,\delta}(\vec{B}^*)}\leqslant C\|a\|_{X'(\vec{A})}. \tag{13}$$

It is not hard to verify that $X_{\kappa,\delta}{}' = (X_{\kappa_1,\delta})^+$, where $\kappa_1(t) = \delta'(t)/\kappa(t)$, and that $X_{\kappa_1,\delta}$ is interpolative between L_1 and L_∞. Then it follows from (13) by Lemma 2 that

$$\|Ta\|_{\left(X_{\varkappa_1,\delta}(\vec{B}^t)\right)^*}\leqslant C\|a\|_{X'(\vec{A})}$$

or $|R(a,b)|\leqslant C\|a\|_{X'(\vec{A})}\|b\|_{X_{\varkappa_1,\delta}(\vec{B}^t)}$. The theorem is proved.

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DICHOTOMY OF THE SPECTRUM OF NON-SELF-ADJOINT OPERATORS

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Let $\mathscr{H}$ be a complex Hilbert space and let A, B be two self-adjoint operators from the Banach algebra End $\mathscr{H}$ of bounded linear operators, acting in $\mathscr{H}$. We assume that the operator B is (continuously) invertible. If the operators A and B commute, then the operator A + iB is normal and, therefore, its spectrum $\sigma(A + iB)$ is contained in the set $\{\lambda + i\mu \in \mathbf{C}: \lambda \in \sigma(A),\ \mu \in \sigma(B)\}$. Consequently, the spectrum of the operator A + iB admits dichotomy (the terminology is taken from [1]), i.e., it does not intersect the imaginary axis $i\mathbf{R}$. If, however, the operators A and B do not commute, then, in general, such a result does not hold. However, under the additional condition $\|A\|\,\mathrm{dist}(\{0\}, \sigma(B)) < 1$ ($\mathrm{dist}(\sigma_1, \sigma_2)$ denotes the distance between the sets σ_1 and σ_2 from $\mathbf{C}$), the spectrum of the operator A + iB does admit dichotomy.

In this paper, as an additional condition, ensuring the dichotomy of the spectrum of the operator A + iB, there occurs, not the condition of the smallness of the norm of the operator A (typical condition of the regular theory of perturbations [2]), but the condition of the smallness of the operator AB — BA (or of other commutators). Since, in this case, the norm of the operator A is not used directly, it is natural to consider also unbounded operators A; therefore, the results obtained here refer also to the theory of singular perturbations [3], as a consequence of which there arises the possibility of applications to differential operators with slowly varying "coefficients."

A Theorem on the Dichotomy of the Spectrum

In this paper we consider operators of the form

$$L = A - B\colon D(L) = D(A) \subset \mathscr{H} \to \mathscr{H},$$

where $A\colon D(A) \subset \mathscr{H} \to \mathscr{H}$ is a skew-adjoint (i.e., iA is self-adjoint) operator with domain of definition D(A) from $\mathscr{H}$, while B is a self-adjoint operator from End $\mathscr{H}$. For the formulation of the results, precisely such a notation of the operators is convenient.

We introduce quantities and concepts, in terms of which the fundamental results will be formulated.

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