ONE PROPERTY OF FUNCTORS OF THE REAL INTERPOLATION METHOD

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1. Introduction

We recall some definitions from interpolation theory of linear operators (for details see [1] or [2]).

A Banach space X is said to be intermediate for a Banach pair $\vec{X} = (X_0, X_1)$, if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ (henceforth, the embeddings are continuous); here, the collection (X_0, X_1, X) is called a triple. We say that a triple (X_0, X_1, X) is interpolative relative to a triple (Y_0, Y_1, Y), if for each linear operator T: $X_i \to Y_i$ $(i = 0, 1)$ $T(X) \subset Y$. If X = Y, X is said to be an interpolative space between the pairs (X_0, X_1) and (Y_0, Y_1) (if, moreover, $X_i = Y_i$ $(i = 0, 1)$, then X is an interpolative space between the spaces X_0 and X_1). In particular, for each interpolative functor (i.f.) (see [1] or [2]) F and arbitrary Banach pairs $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ the triple $(X_0, X_1, F(\vec{X}))$ If, in addition, for each linear operator $(Y_0, Y_1, F(\vec{Y}))$

$$\| T \|_{F(\vec{X}) \to F(\vec{Y})} \leqslant \max_{i=0,1} (\| T \|_{X_i \to Y_i}),$$

then F is said to be an exact i.f.

For a Banach pair (X_0, X_1) and $0 < s, t < \infty$, we define Peetre's $\mathscr{K}$- and $\mathscr{J}$-functionals [3]:

$$\mathscr{K}(s, t, x; X_0, X_1) = \inf_{\substack{x = x_0 + x_1 \\ x_i \in X_i}} \{ s \| x_0 \|_{X_0} + t \| x_1 \|_{X_1} \} \quad (x \in X_0 + X_1);$$

$$\mathscr{J}(s, t, x; X_0, X_1) = \max \{ s \cdot \| x \|_{X_0}, \ t \cdot \| x \|_{X_1} \} \quad (x \in X_0 \cap X_1)$$

If s = 1, these functionals are denoted by $\mathscr{K}(t, x; X_0, X_1)$ and $\mathscr{J}(t, x; X_0, X_1)$.

Let E be a Banach ideal space (BIS) (e.g., see [4]) of two-sided number sequences $a = (a_j)_{j=-\infty}^{\infty}$. We denote by $(X_0, X_1)_E^{\mathscr{K}}$ the set of all $x \in X_0 + X_1$, for which the norm $\| x \| = \| (\mathscr{K}(2^j, x; X_0, X_1))_j \|_E$; is finite; by $(X_0, X_1)_E^{\mathscr{J}}$ the set of all $x \in X_0 + X_1$, admitting a representation

$$x = \sum_{j=-\infty}^{\infty} u_j \text{ (convergence in } X_0 + X_1), \tag{1}$$

where $u_j \in X_0 \cap X_1$. The norm in $(X_0, X_1)_E^{\mathscr{J}}$ is defined as $\inf \| (\mathscr{J}(2^j, u_j; X_0, X_1))_j \|_E$ over all (u_j) in (1).

For a BIS of sequences G and a function f, we denote by G(f) the space with the norm $\| (a_k) \|_{G(f)} = \| (a_k \cdot | f(2^k) |) \|_G$. If $l_\infty(\max(1, 1/t)) \subset E \subset l_1(\min(1, 1/t))$, then the map $(X_0, X_1) \mapsto (X_0, X_1)_E^{\mathscr{K}}$ (respectively, $(X_0, X_1) \mapsto (X_0, X_1)_E^{\mathscr{J}}$) determines an exact i.f.; their collection is called the $\mathscr{K}$method (respectively, the $\mathscr{J}$-method) of interpolation. An important special case (e.g., see [1]) involves the spaces:

$$(X_0, X_1)_{\theta, p} = (X_0, X_1)^{\mathscr{K}}_{l_p(t^{-\theta})} = (X_0, X_1)^{\mathscr{J}}_{l_p(t^{-\theta})} \quad (0 < \theta < 1, \quad 1 \leqslant p \leqslant \infty).$$

Suppose now that a Banach space A is intermediate for a Banach pair $\vec{A} = (A_0, A_1)$. We define exact i.f. $\mathcal{O}_A^{\vec{A}}$ (the orbit of A relative to $\vec{A}$) and $\mathscr{P}_A^{\vec{A}}$ (the coorbit of A relative to $\vec{A}$) as follows (see [5]):

$$\mathcal{O}_A^{\vec{A}}(X_0, X_1) = \{ \sum_{k=1}^{\infty} T_k a_k : \quad \sum_{k=1}^{\infty} \max_{i=0,1} \| T_k \|_{A_i \to X_i} \cdot \| a_k \|_A < \infty \}$$

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with the norm $\inf\{\sum_{k=1}^{\infty}\max_{i=0,1}\|T_k\|_{A_i\to X_i}\cdot\|a_k\|_A\colon x=\sum_{k=1}^{\infty}T_ka_k\}$; $\mathcal{P}_A^{\vec{A}}(X_0, X_1)$ is the set of all $x\in X_0+X_1$ such that for each linear operator $T\colon X_i\to A_i$ $(i=0,1)$ $Tx\in A$ with the norm $\sup\{\|Tx\|_A\colon \max_{i=0,1}\|T\|_{X_i\to A_i}\leqslant 1\}$.

We introduce a partial order on the set of i.f.: $F\leqslant G$, if for each Banach pair (X_0, X_1) $F(X_0, X_1)\subset G(X_0, X_1)$. If $F\leqslant G$ and $G\leqslant F$ we write: F = G. Under this order $\mathcal{O}_A^{\vec{A}}$ is the least i.f. with the property $F(A_0, A_1)\supset A$, and $\mathcal{P}_A^{\vec{A}}$ the greatest i.f. with the property $F(A_0, A_1)\subset A$ [5].

For a positive function f on (0, ∞), we define the stretching function $\mathcal{M}_f$:

$$\mathcal{M}_f(t)=\sup_{s>0}(f(st)/f(s))\qquad (0<t<\infty).$$

Since $\mathcal{M}_f$ is semimultiplicative, there exist numbers

$$\gamma_f=\lim_{t\to 0+}\frac{\ln\mathcal{M}_f(t)}{\ln t}\ \text{и}\ \delta_f=\lim_{t\to\infty}\frac{\ln\mathcal{M}_f(t)}{\ln t}.$$

the lower and upper stretching exponents of the function f, respectively ([2], p. 75). If f is a quasiconcave function on (0, ∞) (i.e., f(t) is increasing and $t^{-1}f(t)$ is decreasing), then $0\leqslant\gamma_f\leqslant\delta_f\leqslant 1$.

We introduce a fundamental definition. A set of Banach pairs $\mathfrak{B}$ is said to be sufficient for an i.f. F if the condition that an i.f. G coincides with F on $\mathfrak{B}$ implies that G = F.

Henceforth, we use the following notation. If f_0, f_1 and h are positive functions on (0, ∞), then $h^{-1}=1/h$, $\tilde{h}(t)=t/h(t)$, $f_h=f_0\cdot h(f_1\cdot f_0^{-1})$, $\overrightarrow{l_p(f^{-1})}$ is the Banach pair $(l_p(f_0^{-1}), l_p(f_1^{-1}))$ $(1\leqslant p\leqslant\infty)$. Suppose that f_0 and f_1 are quasiconcave, $\gamma_{f_1\cdot f_0^{-1}}>0$ and $0<\gamma_h\leqslant\delta_h<1$. Then the results of [6] immediately imply that the set consisting only of two pairs, $\overrightarrow{l_1(f^{-1})}$ and $\overrightarrow{l_\infty(f^{-1})}$, is sufficient for the i.f. $(\cdot,\cdot)^{\mathcal{K}}_{l_p(h^{-1})}$ $(1\leqslant p\leqslant\infty)$. In Sec. 2 a similar result will be obtained for more general i.f. of the $\mathcal{K}$-method in whose parameter l_p is replaced by an arbitrary symmetric space of sequences. At the same time, we will prove a new interpolative relation (see (16)), special cases of which have been known (see [1, p. 157] and [6]).

There also exist other two-element sets of pairs sufficient for i.f. of the real interpolation method. In Sec. 3, the case of symmetric functional spaces is studied (see [2, Chap. 2]). In that situation, the l_1- and l_∞-spaces with "weights" are replaced by the spaces of Lorentz and Marcinkiewicz. Let f be a positive concave function on (0, ∞), f(0) = 0. The Lorentz space Λ(f) is the Banach space with the norm

$$\|x\|_{\Lambda(f)}=\int_0^\infty x^*(t)\,df(t),$$

where x*(t) is the decreasing permutation of |x(t)|. In the Marcinkiewicz space M(f) the norm is defined as follows:

$$\|x\|_{M(f)}=\sup_{t>0}\left(\int_0^t x^*(s)\,ds/f(t)\right).$$

As usual, we call functions f and g equivalent (denoted by f ~ g) if for some C > 0 and all 0 < t < ∞

$$C^{-1}\,|f(t)|\leqslant|g(t)|\leqslant C\,|f(t)|.$$

The equivalence of sequences is defined in a similar way. Finally, $\{e_k\}_{k=-\infty}^{\infty}$ denotes, henceforth, the standard basis in spaces of two-sided sequences.

2. Spaces of Sequences

THEOREM 1. Let f_0 and f_1 be quasiconcave functions on (0, ∞) such that the relation $f_1\cdot f_0^{-1}$ is increasing. If E is a BIS interpolative between $\overrightarrow{l_1(f^{-1})}$ and $\overrightarrow{l_\infty(f^{-1})}$, then the only i.f. (up to equality defined in Sec. 1) satisfying the relation

$$F(\overrightarrow{l_1(f^{-1})})=F(\overrightarrow{l_\infty(f^{-1})})=E,\tag{2}$$

is the exact i.f. $F_{f_0,f_1,E}$:

$$F_{f_0, f_1, E}(X_0, X_1) = \mathcal{O}_E^{\overrightarrow{l_1(f^{-1})}}(X_0, X_1) = \mathcal{P}_E^{\overrightarrow{l_\infty(f^{-1})}}(X_0, X_1) =$$
$$= \{x \in X_0 + X_1 : \| (\mathcal{K}(f_0(2^k), f_1(2^k), x; X_0, X_1))_k \|_E < \infty\} =$$
$$= \{x = \sum_{k=-\infty}^{\infty} u_k : \| (\mathcal{J}(f_0(2^k), f_1(2^k), u_k; X_0, X_1))_k \|_E < \infty\}.$$

Proof. First of all, note that, by the assumption on the existence of a space interpolative between $\overrightarrow{l_1(f^{-1})}$ and $\overrightarrow{l_\infty(f^{-1})}$, and by Theorem 2 in [7], $\gamma_{f_1 \cdot f_0^{-1}} > 0$, and, thus, the relation $f_1 \cdot f_0^{-1}$ is increasing from 0 to ∞.

We denote the functors $\mathcal{O}_E^{\overrightarrow{l_1(f^{-1})}}$ and $\mathcal{P}_E^{\overrightarrow{l_\infty(f^{-1})}}$ by G and H, respectively. Since the space E is interpolative between $l_\infty(f_0^{-1})$ and $l_\infty(f_1^{-1})$ and between $l_1(f_0^{-1})$ and $l_1(f_1^{-1})$, we have (see [2, p. 56])

$$G(\overrightarrow{l_1(f^{-1})}) = H(\overrightarrow{l_\infty(f^{-1})}) = E. \tag{3}$$

Our main goal is proving the equality G = H; to this end, we will need several auxiliary statements.

For each t > 0 we denote by $t\mathbf{R}$ the space of real numbers with the norm $\| a \|_{t\mathbf{R}} = t \cdot | a |$. If F is an exact i.f., then its characteristic function is defined by the isometric equality:

$$F(\mathbf{R}, 1/t\,\mathbf{R}) = (1/g(t))\mathbf{R}. \tag{4}$$

LEMMA 1. Suppose that F is an exact i.f. with the characteristic function g and, for some $1 \leqslant p_0, p_1 \leqslant \infty$, $F(l_{p_0}(f_0^{-1}), l_{p_1}(f_1^{-1})) = E$. Then the sequences $(\| e_k \|_E)$ and $(f_g^{-1}(2^k))$ are equivalent.

Proof. For each $k \in \mathbf{Z}$ we define operators $T_k(a_j) = a_k$ and $S_k a = a \cdot e_k$ $(a \in \mathbf{R})$. Since

$$\| T_k \|_{l_1(f_i^{-1}) \to (f_i^{-1}(2^k))\mathbf{R}} = 1$$

and (see (4))

$$F((f_0^{-1}(2^k))\mathbf{R}, (f_1^{-1}(2^k))\mathbf{R}) = (f_g^{-1}(2^k)) \cdot \mathbf{R}, \tag{5}$$

we deduce, by the exactness of the i.f., that

$$\| T_k \|_{E \to (f_g^{-1}(2^k))\mathbf{R}} \leqslant C_1, \tag{6}$$

where $C_1 > 0$ does not depend on $k \in \mathbf{Z}$.

Similarly, by virtue of the fact that

$$\| S_k \|_{(f_i^{-1}(2^k))\mathbf{R} \to l_1(f_i^{-1})} = 1,$$

and by relationship (5), we obtain:

$$\| S_k \|_{(f_g^{-1}(2^k))\mathbf{R} \to E} \leqslant C_2, \tag{7}$$

where $C_2 > 0$ also does not depend on $k \in \mathbf{Z}$. The statement of the lemma follows from (6) and (7).

We recall some notation from [7]. Suppose the shift operator $P_k(a_j) = (a_{j+k})_{j=-\infty}^{\infty}$ is continuous in the BIS E for all $k \in \mathbf{Z}$. Then there exist numbers

$$\mu_E = \lim_{k \to -\infty} \frac{\ln \| P_k \|_{E \to E}}{k}, \quad \nu_E = \lim_{k \to \infty} \frac{\ln \| P_k \|_{E \to E}}{k}.$$

In the next three statements f_0, f_1 and E are like those in Theorem 1.

LEMMA 2. If $(\| e_k \|_E)_k \sim (g^{-1}(2^k))_k$, then $\gamma_{g f_0^{-1}} > 0$ and $\delta_{g f_1^{-1}} < 1$ (g is a positive function on $(0, \infty)$).

Proof. It suffices to show that for some C > 0 and all $k \in \mathbf{Z}$

$$\mathcal{M}_{g f_0^{-1}}(2^{-k}) \leqslant C \cdot \| P_{-k} \|_{E(f_0) \to E(f_0)} \tag{8}$$

and

$$\mathcal{M}_{g f_1^{-1}}(2^k) \leqslant C \| P_k \|_{E(f_1^{-1}) \to E(f_1^{-1})}. \tag{9}$$

Indeed, then, by the fact that E is interpolative and by Theorem 1 in [7], $\gamma_{g/\tilde{f}_0^{-1}} \geqslant \mu_{E(f_0)} > 0$ and $\delta_{g/f_1} \leqslant \nu_{E(\tilde{f}_1^{-1})} < 1$.

For each k = 0, 1, . . . we find j_k for which

$$\frac{g(2^{-k+j_k})\cdot f_0(2^{j_k})}{f_0(2^{-k+j_k})\cdot g(2^{j_k})} \geqslant \frac{1}{2}\mathcal{M}_{g/f_0^{-1}}(2^{-k}).$$

Then, by hypotheses,

$$\| P_{-k}e_{-k+j_k} \|_{E(f_0)} = \| e_{j_k} \|_{E(f_0)} \geqslant \frac{1}{C_1}\frac{f_0(2^{j_k})}{g(2^{j_k})} \geqslant \frac{1}{2C_1}\cdot\frac{f_0(2^{-k+j_k})}{g(2^{-k+j_k})}\mathcal{M}_{g/f_0^{-1}}(2^{-k}) \geqslant \frac{1}{2C_1}\mathcal{M}_{g/f_0^{-1}}(2^{-k})\, \| e_{-k+j_k} \|_{E(f_0)},$$

whence

$$\mathcal{M}_{g/f_0^{-1}}(2^{-k}) \leqslant 2\cdot C_1^2\cdot \sup_{m\in\mathbb{Z}} \frac{\| P_{-k}e_m \|_{E(f_0)}}{\| e_m \|_{E(f_0)}} \leqslant 2\cdot C_1^2\cdot \| P_{-k} \|_{E(f_0)\to E(f_0)},$$

and, therefore, (8) is proved. The proof of (9) is similar.

The following two statements are given without proofs. In the case when E is l_p with "weight," the former has been proved in [6] (see Theorem 13); the extension to the general case presents no difficulties. The proof of the latter statement generalizing the fundamental lemma of interpolation theory is essentially identical to the proof of that lemma itself (e.g., see [1, p. 62]).

Proposition 1. For each Banach pair $\vec{X} = (X_0, X_1)$ hold the equalities

$$\| x \|_{G(\vec{X})} = \inf\{\| (\mathcal{J}(f_0(2^k), f_1(2^k), u_k; X_0, X_1))_k \|_E : x = \textstyle\sum_{k=-\infty}^{\infty} u_k\}$$

and

$$\| x \|_{H(\vec{X})} = \| (\mathcal{K}(f_0(2^k), f_1(2^k), x; X_0, X_1))_k \|_E.$$

LEMMA 3. Suppose that for some $x \in X_0 + X_1$

$$\lim_{k\to\pm\infty} [\min_{i=0,1} (f_i^{-1}(2^k))\cdot\mathcal{K}(f_0(2^k), f_1(2^k), x; X_0, X_1)] = 0. \tag{10}$$

Then for each $\varepsilon > 0$ there exists a representation $x = \sum_{k=-\infty}^{\infty} u_k$ such that for $k \in \mathbb{Z}$

$$\mathcal{J}(f_0(2^k), f_1(2^k), u_k; X_0, X_1) \leqslant (\gamma + \varepsilon)\quad \mathcal{K}(f_0(2^k), f_1(2^k), x; X_0, X_1),$$

where $\gamma \leqslant 3$ is a universal constant.

We continue the proof of Theorem 1. We will first show that each Banach pair (X_0, X_1) and each $x \in H$ (X_0, X_1) satisfy (10).

Let g_H be the characteristic function of the i.f. H. Then, by (3), [8, p. 48], and also [9, p. 23],

$$E = H(\overrightarrow{l_\infty(f^{-1})}) \subset (\overrightarrow{l_\infty(f^{-1})})^{\mathcal{K}}_{l_\infty(g_H^{-1})} = l_\infty(f_{g_H}^{-1}),$$

so, by Proposition 1, for each $x \in H(X_0, X_1)$ there exists C > 0 such that

$$\mathcal{K}(f_0(2^k), f_1(2^k), x; X_0, X_1) \leqslant C\cdot f_{g_H}(2^k). \tag{11}$$

On the other hand, (3) and Lemma 1 imply the equivalence of the sequences $(\| e_k \|_E)$ and $(f_{g_H}(2^k))$, whence, by Lemma 2 and Lemma 3 in [7], $0 < \gamma_{g_H} \leqslant \delta_{g_H} < 1$. Thus, in particular,

$$\lim_{\substack{t\to 0\\ t\to\infty}} [\min(1, 1/t)\, g_H(t)] = 0. \tag{12}$$

From (11), (12), and the fact that the relation $f_1\cdot f_0^{-1}$ is increasing from 0 to ∞, follows the validity of (10) for $x \in H(X_0, X_1)$.

Everything is now ready for the proof of the equality G = H. Suppose a linear operator T: $l_1(f_i^{-1}) \to l_\infty(f_i^{-1})$ $(i = 0, 1)$ and $a \in E$. Since E is interpolative between $\overrightarrow{l_1(f^{-1})}$ and $\overrightarrow{l_\infty(f^{-1})}$, $Ta \in E$, and, thus, $H(\overrightarrow{l_1(f^{-1})}) \supset E$. Since G is the least i.f. possessing this property, we obtain: $G \leqslant H$. Conversely, let $x \in H(X_0, X_1)$. Then (10) holds and, by Lemma 3, for each $\varepsilon > 0$ there exists a representation $x = \sum_{k=-\infty}^{\infty} u_k$ such that for $k \in \mathbf{Z}$

$$\mathcal{J}(f_0(2^k), f_1(2^k), u_k; X_0, X_1) \leqslant (\gamma + \varepsilon) \mathcal{K}(f_0(2^k), f_1(2^k), x; X_0, X_1).$$

Therefore, by Proposition 1, $x \in G(X_0, X_1)$ and $\|x\|_{G(X_0, X_1)} \leqslant (\gamma + \varepsilon) \|x\|_{H(X_0, X_1)}$.

Thus, G = H. We denote the i.f. G by $F_{f_0, f_1, E}$. By definition, $F_{f_0, f_1, E}$ possesses the required properties. We will prove its uniqueness. Suppose that for some i.f. F_1 $F_1(\overrightarrow{l_1(f^{-1})}) = F_1(\overrightarrow{l_\infty(f^{-1})}) = E$. Then, since G and H are extremal, $G \leqslant F_1$ and $F_1 \leqslant H$, whence $F_1 = F_{f_0, f_1, E}$. Theorem 1 is proved.

The functor $F_{f_0, f_1, E}$ constructed in Theorem 1 depends, in general, on three parameters: f_0, f_1, and E. Under certain assumptions on E, this dependence is substantially simplified. Recall that a BIS of sequences E is said to be symmetric if for each injective map $\pi: \mathbf{Z} \to \mathbf{Z}$, $a = (a_j)_{j=-\infty}^{\infty} \in E$ and

$$\|a_\pi\|_E = \|a\|_E \quad (a_\pi = (a_{\pi(j)})_{j=-\infty}^{\infty}).$$

THEOREM 2. Suppose that E is a symmetric space of sequences, f_0, f_1, h_0, h_1, g are quasiconcave functions on $(0, \infty)$, the functions $f_1 \cdot f_0^{-1}$ and $h_1 \cdot h_0^{-1}$ are increasing, $\gamma_{f_1 \cdot f_0^{-1}} > 0$, $\gamma_{h_1 \cdot h_0^{-1}} > 0$ and $0 < \gamma_g \leqslant \delta_g < 1$. Then $F_{f_0, f_1, E_f} = F_{h_0, h_1, E_h}$, where $E_f = E(f_g^{-1})$ and $E_h = E(h_g^{-1})$.

Two lemmas are necessary for the proof, the first of which is proved in a standard manner.

LEMMA 4. Let (E_0, E_1, E) be a triple of BIS of sequences, v a positive function on $(0, \infty)$. Then $\mathcal{O}_E^{\vec{E}} = \mathcal{O}_{E(v)}^{\overrightarrow{E(v)}}$, where $\vec{E} = (E_0, E_1)$, $\overrightarrow{E(v)} = (E_0(v), E_1(v))$.

LEMMA 5. Suppose that functions f_0, f_1, g and $w = f_1 \cdot f_0^{-1}$ satisfy the hypotheses of Theorem 2. For A > 0, B > 0, we define a set $V(A, B) \subset \mathbf{Z}$ by

$$V(A, B) = \{k\colon A \leqslant g(w(2^k)) \leqslant B\}.$$

Then for each $C_1 > 0$ there exists $C_2 > 0$ such that $B \cdot A^{-1} \leqslant C_1$ implies card $V(A, B) \leqslant C_2$.

Proof. Since $\gamma_w > 0$ and $\gamma_g > 0$, то $\lim_{t \to 0+} w(t) = \lim_{t \to \infty} \frac{1}{w(t)} = \lim_{t \to 0+} g(t) = \lim_{t \to \infty} \frac{1}{g(t)} = 0$. Therefore, for any A > 0 and B > 0 there exist n = 0, 12, . . . and m = 0, 1, . . ., such that

$$g(w(2^n)) \leqslant A \leqslant g(w(2^{n+1})),$$

$$g(w(2^{n+m-1})) \leqslant B \leqslant g(w(2^{n+m})).$$

Clearly, card $V(A, B) \leqslant m + 1$, and if $BA^{-1} \leqslant C_1$, then

$$\frac{g(w(2^{n+m}))}{g(w(2^n))} \leqslant 4 \frac{B}{A} \leqslant 4C_1.$$

By the hypotheses of the lemma, for some $\varepsilon > 0$ and $k \in \mathbf{Z}$

$$\frac{g(w(2^k))}{g(w(2^k))} \leqslant \frac{g(\mathcal{M}_w(1/2)\, w(2^{k+1}))}{g(w(2^{k+1}))} \leqslant \mathcal{M}_g\left(\mathcal{M}_w\left(\frac{1}{2}\right)\right) \leqslant 2^{-\varepsilon}.$$

So

$$2^{m\varepsilon} \leqslant \frac{g(w(2^{n+m}))}{g(w(2^n))} \leqslant 4 \cdot C_1,$$

whence

$$\text{card } V(A, B) \leqslant m + 1 \leqslant \varepsilon^{-1} \cdot \log_2(4C_1) + 1 = C_2.$$

Proof of Theorem 2. By Theorem 3 in [7], E_f is interpolative between $\overrightarrow{l_1(f^{-1})}$ and $\overrightarrow{l_\infty(f^{-1})}$ (respectively, E_h between $\overrightarrow{l_1(h^{-1})}$ and $\overrightarrow{l_\infty(h^{-1})}$). Furthermore, by Theorem 1 and Lemma 4, F_{f_0, f_1, E_f} is the least among i.f. F satisfying the condition

$$F\left(l_1(g(f_1 \cdot f_0^{-1})), l_1\left(\frac{g(f_1 \cdot f_0^{-1})}{f_1 \cdot f_0^{-1}}\right)\right) \supset E.$$

A similar statement is true for F_{h_0, h_1, E_h}. So for the proof of the theorem it suffices to show the validity of the inclusion

$$F_{h_0, h_1, E_h}\left(l_1(g(f_1\cdot f_0^{-1})), l_1\left(\frac{g(f_1\cdot f_0^{-1})}{f_1\cdot f_0^{-1}}\right)\right)\supset E, \tag{13}$$

and then use the extremality of the i.f. F_{f_0, f_1, E_f} and equal status of f_i and h_i.

Since $\gamma_{h_1\cdot h_0^{-1}} > 0$, we have

$$\lim_{t\to 0+}\frac{h_1(t)}{h_0(t)} = \lim_{t\to\infty}\frac{h_0(t)}{h_1(t)} = 0,$$

so for each $k\in\mathbf{Z}$ there exists

$$j_k = \min\left\{j: \frac{f_1(2^k)}{f_0(2^k)} \leqslant \frac{h_1(2^{j_k})}{h_0(2^{j_k})}\right\}.$$

Next, it follows from the definition of j_k and quasiconcavity of the functions $h_1\cdot h_0^{-1}$ and g that

$$\frac{1}{2}g\left(\frac{h_1(2^{j_k})}{h_0(2^{j_k})}\right)\leqslant g\left(\frac{f_1(2^k)}{f_0(2^k)}\right)\leqslant g\left(\frac{h_1(2^{j_k})}{h_0(2^{j_k})}\right) \tag{14}$$

and

$$\frac{h_0(2^{j_k})}{h_1(2^{j_k})}g\left(\frac{h_1(2^{j_k})}{h_0(2^{j_k})}\right)\leqslant \frac{f_0(2^k)}{f_1(2^k)}g\left(\frac{f_1(2^k)}{f_0(2^k)}\right)\leqslant 2\cdot\frac{h_0(2^{j_k})}{h_1(2^{j_k})}g\left(\frac{h_1(2^{j_k})}{h_0(2^{j_k})}\right). \tag{15}$$

Let $N_j = \operatorname{card}\{k: j_k = j\}$. We will prove that $N = \sup_{j\in\mathbf{Z}} N_j < \infty$. To this end, we fix j and denote

$$A = g\left(\frac{h_1(2^j)}{h_0(2^j)}\right).$$

Then, if k is such that $j_k = j$, then, by virtue of (14),

$$\frac{1}{2}A\leqslant g\left(\frac{f_1(2^k)}{f_0(2^k)}\right)\leqslant A.$$

So, by Lemma 5, in its notation,

$$N_j\leqslant N\left(\frac{1}{2}A, A\right)\leqslant C,$$

where $C > 0$ does not depend on A and, therefore, on $j\in\mathbf{Z}$. Thus, $N < \infty$.

Now, we define for each $a = (a_k)\in E$ a sequence $a' = (a_j')_{j=-\infty}^{\infty}$:

$$a_j' = \begin{cases} \max\limits_{j_k=j}|a_k|, & j\in V,\\ 0, & j\,\overline{\in}\, V,\end{cases}$$

where V is the set of all $j\in\mathbf{Z}$, for which $j_k = j$ for some $k\in\mathbf{Z}$. Since E is a symmetric space, $a' = (a_j')\in E$ and $\|a'\|_E\leqslant\|a\|_E$. We define a linear operator T:

$$T(z_j) = \left(a_k\cdot\frac{z_{j_k}}{a_{j_k}'}\right)_{k=-\infty}^{\infty}.$$

Then $T(a_j') = (a_k)$, and T is continuous from $l_1(g(h_1\cdot h_0^{-1}))$ to $l_1(g(f_1\cdot f_0^{-1}))$. Indeed, by (14),

$$\|Tz\|_{l_1(g(f_1/f_0))} = \sum_{k=-\infty}^{\infty}\frac{|a_k z_{j_k}|}{a_{j_k}'}g\left(\frac{f_1(2^k)}{f_0(2^k)}\right)\leqslant\sum_{k=-\infty}^{\infty}\frac{|a_k z_{j_k}|}{a_{j_k}'}g\left(\frac{h_1(2^{j_k})}{h_0(2^{j_k})}\right) =$$

$$= \sum_{j\in V}\frac{|z_j|}{a_j'}\sum_{j_k=j}|a_k|\,g\left(\frac{h_1(2^j)}{h_0(2^j)}\right)\leqslant N\|z\|_{l_1(g(h_1/h_0))}.$$

In exactly the same way but using the relationship (15) one proves that

$$\| Tz\|_{l_1\left(\frac{f_0}{f_1} g\left(\frac{f_1}{f_0}\right)\right)} \leqslant 2\cdot N \| z\|_{l_1\left(\frac{h_0}{h_1} g\left(\frac{h_1}{h_0}\right)\right)}.$$

Since the i.f. F_{h_0, h_1, E_h} is the orbit of E relative to the pair $\left(l_1(g(h_1\cdot h_0^{-1})), l_1\left(\frac{g(h_1\cdot h_0^{-1})}{h_1\cdot h_0^{-1}}\right)\right)$ (see Theorem 1 and Lemma 4), the relationship (13) and with it, as has already been noted, also Theorem 2 are proved.

COROLLARY 1. Suppose that E is a symmetric BIS, f_0, f_1, and g are quasiconcave functions on $(0, \infty)$, $f_1\cdot f_0^{-1}$ is increasing, $\gamma_{f_1\cdot f_0^{-1}} > 0$, $0 < \gamma_g \leqslant \delta_g < 1$. Then for each Banach pair (X_0, X_1)

$$F_{f_0, f_1, E(f_g^{-1})}(X_0, X_1) = (X_0, X_1)^{\mathscr{K}}_{E(g^{-1})} = (X_0, X_1)^{\mathscr{J}}_{E(g^{-1})}.$$

COROLLARY 2. In the hypotheses of Corollary 1, for all $1 \leqslant q_0, q_1 \leqslant \infty$

$$(l_{q_0}(f_0^{-1}), l_{q_1}(f_1^{-1}))^{\mathscr{K}}_{E(g^{-1})} = (l_{q_0}(f_0^{-1}), l_{q_1}(f_1^{-1}))^{\mathscr{J}}_{E(g^{-1})} = E(f_g^{-1}). \tag{16}$$

Theorems 1 and 2 immediately imply

THEOREM 3. Suppose that f_0, f_1, and g are quasiconcave functions on $(0, \infty)$, $f_1\cdot f_0^{-1}$ is increasing, $\gamma_{f_1\cdot f_0^{-1}} > 0$, $0 < \gamma_h \leqslant \delta_h < 1$; E is a symmetric BIS of sequences. Then the set $\{\overrightarrow{l_1(f^{-1})}, \overrightarrow{l_\infty(f^{-1})}\}$ is sufficient for the i.f. $(\cdot, \cdot)^{\mathscr{K}}_{E(g^{-1})} = (\cdot, \cdot)^{\mathscr{J}}_{E(g^{-1})}$.

3. Case of Symmetric Functional Spaces

We denote by $\overrightarrow{\Lambda(f)}$ the Banach pair of Lorentz spaces $(\Lambda(f_0), \Lambda(f_1))$, and by $\overrightarrow{M(\tilde{f})}$ the pair of Marcinkiewicz spaces $(M(\tilde{f}_0), M(\tilde{f}_1))$.

THEOREM 4. Suppose that f_0, f_1, and g are concave functions on $(0, \infty)$, $f_1\cdot f_0^{-1}$ is increasing, $\gamma_{f_1\cdot f_0^{-1}} > 0$, $0 < \gamma_{f_i} \leqslant \delta_{f_i} < 1$ $(i = 0, 1)$, $0 < \gamma_g \leqslant \delta_g < 1$, E is a symmetric BIS of sequences. Then $(\overrightarrow{\Lambda(f)}, \overrightarrow{M(\tilde{f})})$ is a sufficient set of the i.f. $(\cdot, \cdot)^{\mathscr{K}}_{E(g^{-1})}$.

In order to prove the theorem we will need

Proposition 2. Suppose that BIS of sequences E_0 and E_1 are interpolative between l_∞ and $l_\infty(1/t)$ and for each $(a_k) \in E_0 \cup E_1$

$$\lim_{k\to-\infty} a_k = 0.$$

Then the equality $(L_1, L_\infty)^{\mathscr{K}}_{E_0} = (L_1, L_\infty)^{\mathscr{K}}_{E_1}$ implies $E_0 = E_1$.

Proof. Since the conditions are symmetric relative to E_0 and E_1, it suffices to show that $E_0 \subset E_1$.

If $(a_j) \in E_0$, then the sequence (a'_k), where

$$a'_k = \sup_{j\in Z}[\min(1, 2^{k-j}) \cdot |a_j|]$$

also belongs to E_0 (see [10] or [11]), so $\lim_{k\to-\infty} a'_k = 0$. We denote

$$a(t) = \sum_{k=-\infty}^{\infty} a'_k \chi_{[2^k, 2^{k+1})}(t).$$

Since for all $0 < t_1 < t_2 < \infty$

$$a(t_1) \leqslant a(t_2) \qquad \frac{a(t_2)}{t_2} \leqslant \frac{a(t_1)}{t_1},$$

we deduce that, by [2] (p. 69), $a(t)$ is equivalent to its least concave majorant $\bar{a}(t)$, and $\lim_{t\to 0+} \bar{a}(t) = \lim_{k\to-\infty} a'_k = 0$. Therefore, $\bar{a}(t)$ is absolutely continuous on $[0, \infty)$ and, thus, by [2, p. 108],

$$\bar{a}(t) = \int_0^t \bar{a}'(s)\, ds = \mathscr{K}(t, \bar{a}'; L_1, L_\infty).$$

Thus, $(\mathscr{K}(2^k, \bar{a}'; L_1, L_\infty)) \sim (a_k')$, whence $\bar{a}' \in (L_1, L_\infty)^{\mathscr{K}}_{E_0} = (L_1, L_\infty)^{\mathscr{K}}_{E_1}$ and $(a_k') \in E_1$. Since $|a_k| \leqslant a_k'$, we conclude that $(a_k) \in E_1$. The proposition is proved.

Proof of Theorem 4. Thus, suppose that an i.f. F satisfies

$$F(\overrightarrow{\Lambda(f)}) = (\overrightarrow{\Lambda(f)})^{\mathscr{K}}_{E(g^{-1})}, \; F(\overrightarrow{M(f)}) = (\overrightarrow{M(f)})^{\mathscr{K}}_{E(g^{-1})}.$$

By [7, Theorem 3], the spaces $l_1(f_i^{-1})$ and $l_\infty(f_i^{-1})$ are interpolative between the pairs $(l_1, l_1(1/t))$ and $(l_\infty, l_\infty(1/t))$. Furthermore (see [8]),

$$\Lambda(f_i) = (L_1, L_\infty)^{\mathscr{K}}_{l_1(f_i^{-1})}, \quad M(f_i) = (L_1, L_\infty)^{\mathscr{K}}_{l_\infty(f_i^{-1})}.$$

So, by reiteration arguments (see [12, Corollary 3] or [10, Corollary 14.1]) and by Corollary 2, we obtain:

$$(L_1, L_\infty)^{\mathscr{K}}_{F(\overrightarrow{l_1(f^{-1})})} = (L_1, L_\infty)^{\mathscr{K}}_{F(\overrightarrow{l_\infty(f^{-1})})} = (L_1, L_\infty)^{\mathscr{K}}_{E(f_g^{-1})}.$$

A direct verification shows that the spaces $F(\overrightarrow{l_1(f^{-1})})$, $F(\overrightarrow{l_\infty(f^{-1})})$ and $E(f_g^{-1})$ satisfy the hypotheses of Proposition 2 and, thus, $F(\overrightarrow{l_1(f^{-1})}) = F(\overrightarrow{l_\infty(f^{-1})}) = E(f_g^{-1})$. Therefore, by Theorem 3, $F = (\cdot, \cdot)^{\mathscr{K}}_{E(f_g^{-1})}$. Theorem 4 is proved.

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