STABLE INTERPOLATION FUNCTORS

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One of the most important results of the theory of interpolation of linear operators is the reiteration theorem for the real method due to Lions and Peetre (see [1] or [2, pp. 66-70]). From it, in particular, it follows that the value of the interpolation functor $(\cdot,\cdot)_{\theta,p}$ $(0<\theta<1,\ 1\leqslant p\leqslant\infty)$ on a pair of Lorentz spaces $L_{q,r}$ with different first indices is independent up to isomorphism from the second indices. In the present note we give results relating to the clarification of the inner reasons for this phenomenon as well as to the description of the class of interpolation functors having similar properties.

Let the functions g, f_0 and f_1 be positive on the half-axis $(0, \infty)$; let E be the Banach ideal space (BIS) of bilateral sequences of real numbers $a=(a_k)_{k=-\infty}^{\infty}$. Everywhere below g^{-1} will denote the function 1/g, and E(g) will be the space of sequences with "weight" $(g(2)^k)_{k=-\infty}^{\infty}$, i.e.,

$$\|(a_k)\|_{E(g)}=\|(a_k\cdot g(2^k))\|_E,$$

and finally, $\overrightarrow{l_p(f^{-1})}$ will be the Banach pair $(l_p(f_0^{-1}),\ l_p(f_1^{-1}))$ $(1\leqslant p\leqslant\infty)$.

We introduce the basic definition. We shall call the interpolation functor [2] F stable (we write: $F\in C$), if $F(\overrightarrow{l_1(f^{-1})})=F(\overrightarrow{l_\infty(f^{-1})})$ for any pair of functions f_0 and f_1, which are quasiconcave on $(0, \infty)$, such that the function $f_1f_0^{-1}$ is increasing and its lower exponent of dilatation $\gamma_{f_1f_0^{-1}}>0$.

The necessity of the latter condition in the definition of a stable operator will become clear after the following result.

THEOREM 1. Let f_0 and f_1 be positive functions on $(0,\infty)$, $f_1f_0^{-1}$ be increasing. Then the following conditions are equivalent:

1) there exists a BIS E such that the triple $(l_1(f_0^{-1}),\ l_1(f_1^{-1}),\ E)$ is interpolational with respect to the triple $(l_\infty(f_0^{-1}),\ l_\infty(f_1^{-1}),\ E)$ [2];

2) $\gamma_{f_1f_0^{-1}}>0$.

Theorems 2 and 3 are basic. In their proof one makes essential use of the method of interpolational orbits [3, 4].

THEOREM 2. The interpolation functor $F\in C$ if and only if one can find a pair of quasiconcave functions g_0 and g_1, such that $g_1g_0^{-1}$ is increasing and $F(\overrightarrow{l_1(g^{-1})})=F(\overrightarrow{l_\infty(g^{-1})})$.

THEOREM 3. The class of stable interpolation functors coincides with the subclass $\mathscr{K}$ of the method of real interpolation [5], consisting of functors of the form $(\cdot,\cdot)_E^{\mathscr{K}}$, where E is the BIS of sequences in which the Calderon operator Q, $Qa=(Qa)_k$, $(Qa)_k=\sum_{j=-\infty}^{\infty}a_j\min(1,2^{k-j})$ is continuous.

COROLLARY 1. Let G be a symmetric space of sequences [6], h be a quasiconcave function on $(0, \infty)$.

The interpolation functor $(\cdot,\cdot)_{G(h^{-1})}^{\mathscr{K}}$ is stable if and only if the exponents of dilatation of the function h are nontrivial, i.e., $0<\gamma_h\leqslant\delta_h<1$.

Now we give a generalization of the Lions—Peetre iteration theorem mentioned above.

THEOREM 4. Let us assume that f_0 and f_1 are quasiconcave functions on $(0,\infty)$, $f_1f_0^{-1}$ is increasing, and $\gamma_{f_1f_0^{-1}}>0$.

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Then for any stable functor F and any Banach pair (X_0, X_1),

$$F\left((X_0, X_1)^{\mathscr{F}}_{l_1(f_0^{-1})}, (X_0, X_1)^{\mathscr{F}}_{l_1(f_1^{-1})}\right) = F\left((X_0, X_1)^{\mathscr{K}}_{l_\infty(f_0^{-1})}, (X_0, X_1)^{\mathscr{K}}_{l_\infty(f_1^{-1})}\right) = (X_0, X_1)^{\mathscr{K}}_{E},$$

where $E = F(\overrightarrow{l_1(f^{-1})}) = F(\overrightarrow{l_\infty(f^{-1})})$.

COROLLARY 2. Let the functions f_0 and f_1 satisfy the conditions of Theorem 4, G be a symmetric space of sequences, h be a quasiconcave function on $(0, \infty)$, $0 < \gamma_h \leqslant \delta_h < 1$.

Then for any Banach pair (X_0, X_1) one has

$$\left((X_0, X_1)^{\mathscr{F}}_{l_1(f_0^{-1})}, (X_0, X_1)^{\mathscr{F}}_{l_1(f_1^{-1})}\right)^{\mathscr{K}}_{G(h^{-1})} = \left((X_0, X_1)^{\mathscr{K}}_{l_\infty(f_0^{-1})},\right.$$
$$\left.(X_0, X_1)^{\mathscr{K}}_{l_\infty(f_1^{-1})}\right)^{\mathscr{K}}_{G(h^{-1})} = (X_0, X_1)^{\mathscr{K}}_{G(f_h^{-1})},$$

where $f_h = f_0 h(f_1 f_0^{-1})$.

For $G = l_p$ $(1 \leqslant p \leqslant \infty)$ under certain additional conditions (it turns out that this is now clearly superfluous) the last result was proved in [3] and in [7]. In complete form the special case of Corollary 2 discussed is announced in the author's paper [8]. We also note that if $G = l_p$ and f_0, f_1 and h are powers, Corollary 2 becomes the reiteration theorem of Lions–Peetre.

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LITERATURE CITED

1. J.-L. Lions and J. Peetre, Inst. Hautes Etudes Publ. Math., 19, 5-68 (1964).
2. I. Berg and I. Lefstrem, Interpolational Spaces. Introduction [Russian translation], Mir, Moscow (1980).
3. S. Janson, J. Funct. Anal., 44, No. 1, 50-73 (1981).
4. V. I. Ovchinnikov, Functs. Anal. Prilozhen., 10, No. 4, 45-54 (1976).
5. Yu. A. Brudnyi and N. Ya. Kruglyak, Dokl. Akad. Nauk SSSR, 256, No. 1, 14-17 (1981).
6. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, Interpolation of Linear Operators [in Russian], Nauka, Moscow (1978).
7. J. Gustavsson, Math. Scand., 42, No. 2, 289-305 (1978).
8. S. V. Astashkin, Dokl. Akad. Nauk Uzb. SSR, No. 4, 10-11 (1983).

A CHARACTERISTIC PROPERTY OF SIMPLE LIE ALGEBRAS

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1. Let G be a finite-dimensional complex Lie algebra. G is called metrizable if there is on G a nondegenerate symmetric bilinear form B invariant under the associated representation $B([z, x], y) + B(x, [z, y]) = 0$. The class of metrizable Lie algebras is interesting in that it is in a certain sense the next in complexity after the much studied class of semisimple Lie algebras. In this note we study certain properties of the decomposition of an arbitrary metrizable Lie algebra, which is not nilpotent, relative to a fixed Cartan subalgebra. Our investigation follows, in the main, the standard scheme used in the theory of semisimple Lie algebras. It should be noted that part of the results and proofs of this theory are untrue or require refinements, sometimes significant, when we are considering metrizable Lie algebras.

Although here we omit the proofs, all the assertions are arranged in the order needed for the proof of the main results (a theorem and two corollaries). For the structure theory of semisimple Lie algebras we have used the books [1] and [2]; we have also used certain information on metrizable Lie algebras from [3] and [4].

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