An n-tuple $(w_1, \ldots, w_n)$ gives rise to the group F if and only if $(w_1, \ldots, w_n)$ and $(x_1, \ldots, x_n)$ are equivalent with respect to the transformations 1)-4^1). Indeed, if the n-tuple $(w_1, \ldots, w_n)$ gives rise to the group F, then the following chain of transformations is valid:

$$(w_1, \ldots, w_n) \sim (w_1, \ldots, w_n, 1, \ldots, 1) \sim$$

$$\sim (w_1, \ldots, w_n, x_1, \ldots, x_1) \sim (x_1, \ldots, x_n, w_1, \ldots, w_n) \sim (x_1, \ldots, x_n, 1, \ldots, 1) \sim (x_1, \ldots, x_n).$$

Let us observe that transformations of the form 4) are used only in the beginning, and those of the form 4^1) are used only at the end, of the chain of transformations.

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DESCRIPTION OF THE INTERPOLATION SPACES BETWEEN $(l_1(\omega^0), l_1(\omega^1))$ AND $(l_\infty(\omega^0), l_\infty(\omega^1))$

S. V. Astashkin

Let E be the Banach ideal space of the two-sided sequences of real numbers $a = (a_k)_{k=-\infty}^{\infty}$; and $\omega^0 = (\omega_k^0)_{k=-\infty}^{\infty}$ and $\omega^1 = (\omega_k^1)_{k=-\infty}^{\infty}$ be two sequences of nonnegative numbers. It has been shown in [1] that E is an interpolation space between $(l_1(\omega^0), l_1(\omega^1))$ and $(l_\infty(\omega^0), l_\infty(\omega^1))$ if and only if the operator Q such that $Qa = (Qa)_k$, where

$$(Qa)_k = \sum_{j=-\infty}^{\infty} a_j \cdot \min_{i=0,1} \left\{ \frac{\omega_j^i}{\omega_k^i} \right\} \qquad (a = (a_j) \in E) \tag{1}$$

is continuous in this space.

In this note, we obtain a criterion for the continuity of Q in the case $\omega^i = \left(\frac{1}{f_i(2^k)} \right)_{k=-\infty}^{\infty}$ where f_i in a nonnegative concave function on $(0, \infty)$ $(i = 0, 1)$. By the same token, the interpolation spaces between the corresponding couples are effectively described.

<u>1°. Definitions and Notation.</u> Let (A_0, B_0) and (A_1, B_1) be two Banach couples [2]. A Banach space E such that

$$A_i \cap B_i \subset E \subset A_i + B_i \qquad (i = 0, \quad 1),$$

is called an interpolation space between (A_0, B_0) and (A_1, B_1) if each linear operator that is continuous from A_0 into A_1 and from B_0 into B_1 is continuous in E.

For each nonnegative function f on $(0, \infty)$, its magnification function M_f is defined by the equation

$$M_f(t) = \sup_{0<s<\infty} \frac{f(st)}{f(s)} \qquad (0 < t < \infty).$$

The function M_f is semimultiplicative, and, therefore, the numbers

$$\gamma_f = \lim_{t \to 0+} \frac{\ln M_f(t)}{\ln t} \quad \text{and} \quad \delta_f = \lim_{t \to \infty} \frac{\ln M_f(t)}{\ln t}$$

Voronezh State University. Translated from Matematicheskie Zametki, Vol. 35, No. 4, pp. 497-503, April, 1984. Original article submitted October 28, 1982.

0001-4346/84/3534-0261$08.50

exist [3, p. 75] and are called the lower and the upper magnification indices of f. If f is a concave function, then $0 \leqslant \gamma_f \leqslant \delta_f \leqslant 1$ [3, p. 76].

Let $S(\mathbf{Z})$ be the linear space of all two-sided numerical sequences, $E \subset S(\mathbf{Z})$ be a Banach ideal space [4, Chap. 4]. Let us assume that the shift operator $P_k(a_j) = (a_{k+j})_{j=-\infty}^{\infty}$ is continuous in E for arbitrary k = 0, ±1, ... Since the function $h(2^k) = \|P_k\|_{E\to E}$ is semimultiplicative on the set $\{2^k, k = 0, \pm 1, \ldots\}$, the numbers

$$\mu_E = \lim_{k\to-\infty} \frac{\ln\|P_k\|_{E\to E}}{k} \quad \text{and} \quad \nu_E = \lim_{k\to\infty} \frac{\ln\|P_k\|_{E\to E}}{k}$$

exist [3, p. 75] and $-\infty < \mu_E \leqslant \nu_E < \infty$.

If $\mu_E = \nu_E = 0$ (in other words, $\sup\limits_{k=0,\pm1,\ldots} \|P_k\|_{E\to E} < \infty$), then the space E will be said to be strongly shift-invariant.

Let E be a Banach ideal space of sequences and f be a nonnegative function on (0, ∞). Then E(f) is the space with the norm

$$\|(a_j)\|_{E(f)} = \|(a_j \cdot f(2^j))\|_E.$$

Everywhere in the sequel, we will assume that f_0, f_1, and h are nonnegative concave functions on (0, ∞) and $f_1 \cdot f_0^{-1}$ increases from 0 to ∞.

In conclusion, we introduce the following notation:

$$h^{-1} = \frac{1}{h}, \quad \bar{h}(t) = \frac{t}{h(t)}, \quad f_h = f_0 \cdot h(f_1 \cdot f_0^{-1}), \quad \mu_E^i = \mu_{E(f_i)},$$
$$\nu_E^i = \nu_{E(f_i^{-1})} \ (i = 0, 1), \quad \overrightarrow{l_p(f^{-1})} = (l_p(f_0^{-1}), l_p(f_1^{-1})).$$

2°. Formulation of Results. THEOREM 1. The following statements are equivalent:

1) The Banach ideal space E is an interpolation space between $\overrightarrow{l_1(f^{-1})}$ and $\overrightarrow{l_\infty(f^{-1})}$;

2) $\mu_E^0 > 0$, $\nu_E^1 < 1$.

THEOREM 2. There exists an interpolation space between $\overrightarrow{l_1(f^{-1})}$ and $\overrightarrow{l_\infty(f^{-1})}$ if and only if $\gamma_{f_1 \cdot f_0^{-1}} > 0$.

THEOREM 3. Let G be a strongly shift-invariant Banach ideal space and $\gamma_{f_1 \cdot f_0^{-1}} > 0$. Then $G(f_h^{-1})$ is an interpolation space between $\overrightarrow{l_1(f^{-1})}$ and $\overrightarrow{l_\infty(f^{-1})}$ if and only if $0 < \gamma_h \leqslant \delta_h < 1$.

THEOREM 4. If $\gamma_{f_i} = \delta_{f_i}$ $(i = 0, 1)$, then the Banach ideal space E is an interpolation space between $\overrightarrow{l_1(f^{-1})}$ and $\overrightarrow{l_\infty(f^{-1})}$ if and only if $\gamma_{f_0} < \mu_E \leqslant \nu_E < \gamma_{f_1}$.

3°. Proofs. We formulate and prove a series of lemmas.

We establish the validity of the first of them by standard arguments.

LEMMA 1. Let E_0, E_1, F_0, F_1 and G be Banach ideal spaces of sequences and f be a nonnegative function on (0, ∞). Then G is an interpolation space between (E_0, E_1) and (F_0, F_1) if and only if G(f) is an interpolation space between $(E_0(f), E_1(f))$ and $(F_0(f), F_1(f))$.

We call a sequence $(a_j)_{j=-\infty}^{\infty}$ quasiconcave if $a_j > 0$ and $a_j \leqslant a_{j+1} \leqslant 2\cdot a_j$ $(j = 0, \pm 1, \ldots)$.

LEMMA 2. Let E be a Banach ideal space that is an interpolation space between $(l_\infty, l_\infty(t^{-1}))$ and $(l_\infty, l_\infty(t^{-1}))$, and $T: E \to S(\mathbf{Z})$ be a monotonic linear operator (i.e., if $a = (a_j)$, $a_j \geqslant 0$, then $Ta = (Ta)_k$, $(Ta)_k \geqslant 0$). Suppose that there exists a C > 0 such that

$$\|Ta\|_E \leqslant C\|a\|_E$$

for each quasiconcave $a = (a_j) \in E$.

Then there exists a $C_1(E) > 0$ such that

$$\|Ta\|_E \leqslant C_1(E)\cdot C\cdot\|a\|_E$$

for each $a \in E$.

Proof. For each $a = (a_j) \in E$, the sequence $\hat{a} = (\hat{a}_k)$, where

$$\hat{a}_k = \sup_{-\infty<j<\infty} [\min(1, 2^{k-j}) \cdot |a_j|],$$

is quasiconcave and $|a_k| \leqslant \hat{a}_k$ $(k = 0, \pm 1, \ldots)$. Since E is an interpolation space between $(l_\infty, l_\infty(t^{-1}))$ and $(l_\infty, l_\infty(t^{-1}))$ there exists a $C_1(E) > 0$, such that $\|\hat{a}\|_E \leqslant C_1(E) \cdot \|a\|_E$ [5, p. 36].

Thus, by virtue of the monotonicity of T,

$$\|Ta\|_E \leqslant \|T\hat{a}\|_E \leqslant C\|\hat{a}\|_E \leqslant C_1(E) \cdot C \cdot \|a\|_E.$$

LEMMA 3. If $\gamma_{f_1 \cdot f_0^{-1}} > 0$, then the following conditions are equivalent:

a) $$0 < \gamma_h \leqslant \delta_h < 1;$$

b) $$\gamma_{f_h \cdot f_0^{-1}} > 0, \ \delta_{f_h \cdot \bar{f}_1} < 1.$$

Proof. b) → a). Since $f_1 \cdot f_0^{-1}$ increases from 0 to ∞ and $t^{-1} \cdot f_1(t)$ decreases [3, p. 67], for $0 < s \leqslant 1$ we have

$$M_h(s) = \sup_{0<t<\infty} \frac{h\left(s \cdot \frac{f_1(t)}{f_0(t)}\right)}{h\left(\frac{f_1(t)}{f_0(t)}\right)} \leqslant \sup_{0<t<\infty} \frac{h\left(\frac{f_1(st)}{f_0(st)}\right)}{h\left(\frac{f_1(t)}{f_0(t)}\right)} = M_{f_h \cdot f_0^{-1}}(s).$$

Since $\gamma_{f_h \cdot f_0^{-1}} > 0$, it follows that $\gamma_h > 0$.

Using the relations

$$\bar{h}(f_1 \cdot f_0^{-1}) = f_1 \cdot f_h^{-1} \tag{2}$$

and [3, p. 144]

$$\delta_h = 1 - \gamma_{\bar{h}}, \quad \delta_{f_h \cdot \bar{f}_1} = 1 - \gamma_{f_1 \cdot f_h^{-1}}, \tag{3}$$

in exactly the same manner we get $\delta_h < 1$.

a) → b). Since

$$M_{f_h \cdot f_0^{-1}}(s) = \sup_{0<t<\infty} \frac{h\left(\frac{f_1(st)}{f_0(st)}\right)}{h\left(\frac{f_1(t)}{f_0(t)}\right)} \leqslant M_h(M_{f_1 \cdot f_0^{-1}}(s))$$

for $0 < s \leqslant 1$, it follows from the inequalities $\gamma_{f_1 \cdot f_0^{-1}} > 0$ and $\gamma_h > 0$ that $\gamma_{f_h \cdot f_0^{-1}} > 0$. We also prove the second inequality of b) with the help of (2) and (3).

Proof of Theorem 1. 1) → 2). By Lemma 1, $E(f_0)$ is an interpolation space between $(l_1, l_1(f_0 \cdot f_1^{-1}))$ and $(l_\infty, l_\infty(f_0 \cdot f_1^{-1}))$, and, therefore [1], the operator Q from (1) is continuous in $E(f_0)$ for $\omega^0 = (1)_k$ and $\omega^1 = (f_0(2^k)/f_1(2^k))_k$. Since $f_1 \cdot f_0^{-1}$ is an increasing function, it follows that the operator Q_1, such that $Q_1 a = (Q_1 a)_k = \left(\sum_{j \leqslant k} a_j\right)_k$, is continuous in $E(f_0)$.

For a quasiconcave sequence $a = (a_j) \in E(f_0)$, $k = 1, 2, \ldots$, and $j = 0, \pm 1, \ldots$, we have

$$(P_{-k}a)_j = a_{j-k} \leqslant \frac{1}{k} \sum\nolimits_{s=j-k}^{j} a_s \leqslant \frac{1}{k} (Q_1 a)_j,$$

whence

$$\|P_{-k}a\|_{E(f_0)} \leqslant \frac{\|Q_1\|}{k} \|a\|_{E(f_0)}. \tag{4}$$

Since $E(f_0)$ is an interpolation space between $(l_\infty, l_\infty(f_0 \cdot f_1^{-1}))$ and $(l_\infty, l_\infty(f_0 \cdot f_1^{-1}))$ and $l_\infty(f_0 \cdot f_1^{-1})$ is an interpolation space between $(l_\infty, l_\infty(t^{-1}))$ and $(l_\infty, l_\infty(t^{-1}))$ [1], it follows that $E(f_0)$ is an interpolation space between $(l_\infty, l_\infty(t^{-1}))$ and $(l_\infty, l_\infty(t^{-1}))$. Thus, by virtue of (4) and Lemma 2, there exists a constant $C_1(E, f_0)$ such that

$$\|P_{-k}\|_{E(f_0) \to E(f_0)} \leqslant k^{-1} \|Q_1\| C_1(E, f_0).$$

Therefore [3, p. 75], $\mu_E^0 > 0$.

Considering the space $E(\bar{f}_1^{-1})$, the couples $(l_1(\bar{f}_0^{-1}\cdot\bar{f}_1^{-1}), l_1(t^{-1}))$, and $(l_\infty(\bar{f}_0^{-1}\cdot\bar{f}_1^{-1}), l_\infty(t^{-1}))$, and also the operator Q_2 such that $Q_2a = (Q_2a)_k$, where

$$(Q_2a)_k = \sum_{j\geqslant k} 2^{k-j}\cdot a_j,$$

in exactly the same manner we get

$$\| P_k \|_{E(\bar{f}_1^{-1})\to E(\bar{f}_0^{-1})} \leqslant k^{-1}2^k \| Q_2 \| C_1(E, f_1)$$

for k = 1, 2, ... Hence, again with the help of [3, p. 75], we conclude that $\nu_E^1 < 1$.

2) → 1). It is sufficient to show that the operator Q from (1) is continuous in E in the case $\omega^i = (1/f_i(2^k))_k$ [1]. By virtue of the monotonicity of $f_1\cdot f_0^{-1}$ we have $Qa = \sum_{k=-\infty}^{\infty} b^k$, where

$$b^k = \begin{cases} \left(f_0(2^j)\cdot P_k\left(\frac{a_j}{f_0(2^j)}\right)\right)_{j=-\infty}^{\infty}, & k = 0, -1, \ldots, \\ \left(f_1(2^j)\cdot P_k\left(\frac{a_j}{f_1(2^j)}\right)\right)_{j=-\infty}^{\infty}, & k = 1, 2, \ldots \end{cases}$$

Since, by the condition, $\mu_E^0 > 0$ and $\nu_E^1 < 1$, it follows that for k = 0, −1, . . . we have

$$\| b^k \|_E = \left\| P_k\left(\frac{a_j}{f_0(2^j)}\right)\right\|_{E(f_0)} \leqslant 2^{\varepsilon k} \| a \|_E,$$

and for k = 1, 2, . . . we have

$$\| b^k \|_E = 2^{-k} \| P_k (a_j\cdot \bar{f}_1(2^j)) \|_{E(\bar{f}_1^{-1})} \leqslant 2^{-\varepsilon k} \| a \|_E,$$

where $\varepsilon > 0$. Therefore,

$$\| Qa \|_E \leqslant \sum_{k=-\infty}^{\infty} \| b^k \|_E \leqslant C \| a \|_E.$$

Theorem 1 is proved.

Proof of Theorem 2. If $\gamma_{f_1\cdot f_0^{-1}} > 0$, then by Lemma 3,

$$\gamma_{f_h\cdot f_0^{-1}} > 0, \quad \delta_{f_h\cdot \bar{f}_1} < 1, \tag{5}$$

where $0 < \gamma_h \leqslant \delta_h < 1$.

Let G be an arbitrary strongly shift-invariant space. By direct computation, we verify that

$$\mu_E^0 = \gamma_{f_h\cdot f_0^{-1}}, \quad \nu_E^1 = \delta_{f_h\cdot \bar{f}_1}, \tag{6}$$

where $E = G(f_h^{-1})$. Consequently, by virtue of (5) and Theorem 1, E is an interpolation space between $\overrightarrow{l_1(f^{-1})}$ and $\overrightarrow{l_\infty(f^{-1})}$.

We prove the converse. Let E be a Banach ideal space that is an interpolation space between $\overrightarrow{l_1(f^{-1})}$ and $\overrightarrow{l_\infty(f^{-1})}$, and set $e_j = (\ldots, 0, \underbrace{1}_{j}, 0, \ldots)$ $(j = 0, \pm 1, \ldots)$. Since $P_k e_j = e_{j-k}$ $(j, k = 0, \pm 1, \ldots)$ and, by Theorem 1, $\mu_E^0 > 0$ and $\nu_E^1 < 1$, it follows that for k = 0, 1, 2, . . . we have

$$M_{f_1\cdot f_0^{-1}}(2^k) \leqslant 2 \sup_{j=0,\pm 1,\ldots} \frac{f_1(2^{j-k})\cdot f_0(2^j)}{f_0(2^{j-k})\cdot f_1(2^j)} =$$

$$= 2 \sup_{j=0,\pm 1,\ldots} \frac{\| e_j \|_{E(f_0)}\cdot \| e_{j-k} \|_{E(f_1)}}{\| e_{j-k} \|_{E(f_0)}\cdot \| e_j \|_{E(f_1)}} \leqslant 2^{1-k} \sup_{j=0,\pm 1,\ldots} \frac{\| P_{-k} e_{j-k} \|_{E(f_0)}}{\| e_{j-k} \|_{E(f_0)}} \sup_{j=0,\pm 1,\ldots} \frac{\| P_k e_j \|_{E(\bar{f}_1^{-1})}}{\| e_j \|_{E(\bar{f}_1^{-1})}} \leqslant C\cdot 2^{-\varepsilon k},$$

where $\varepsilon > 0$. Hence $\gamma_{f_1\cdot f_0^{-1}} > 0$. Theorem 2 is proved.

Theorem 3 follows immediately from Lemma 3, the relation (6), and Theorem 1.

Example. Let f_0, f_1, and h be nonnegative concave functions on $(0, \infty)$, $\gamma_{f_1 \cdot f_0^{-1}} > 0$, and $1 \leqslant p \leqslant \infty$. It follows from Theorem 3 that $l_p(f_h^{-1})$ is an interpolation space between $\overline{l_1(f^{-1})}$ and $\overline{l_\infty(f^{-1})}$ if and only if $0 < \gamma_h \leqslant \delta_h < 1$.

Theorem 4 follows from Theorem 1 and the following two inequalities, whose validity can be verified directly:

$$\frac{1}{M_{f_0}(2^k)} \| P_{-k} \|_{E(f_0) \to E(f_0)} \leqslant \| P_{-k} \|_{E \to E} \leqslant M_{f_0}(2^{-k}) \| P_{-k} \|_{E(f_0) \to E(f_0)},$$

$$\frac{2^{-k}}{M_{f_1}(2^{-k})} \| P_k \|_{E(\overline{f}_1^{-1}) \to E(\overline{f}_1^{-1})} \leqslant \| P_k \|_{E \to E} \leqslant 2^{-k} M_{f_1}(2^k) \| P_k \|_{E(\overline{f}_1^{-1}) \to E(\overline{f}_1^{-1})} \quad (k = 0, 1, \ldots).$$

In conclusion, the author thanks E. M. Semenov for assistance with the note.

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IMBEDDING THEOREMS FOR SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

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When studying boundary-value problems for nonlinear differential equations of infinite order, we have to consider spaces of infinitely differentiable functions $u(x)$: $G \to C^1$ having the finite energy integral

$$\rho(u) \equiv \sum_{n=0}^{\infty} a_n \| D^n u \|_r^p,$$

where $p \geqslant 1, r \geqslant 1, a_n \geqslant 0, n = 0, 1, \ldots,$ are numbers, and $\| \cdot \|_r$ is the norm in Lebesgue space $L_r(G)$, $G \subset \mathbf{R}$.

Dubinskii discussed theory of spaces

$$W^\infty\{a_n, p, r\}_{(G)} \equiv \{u(x) \in C^\infty(G), \rho(u) < \infty\}$$

e.g., in [1-3]. In particular, he stated in [3] criteria for the imbedding and compact imbedding of the spaces in terms of the asymptotic behavior of the norms of the imbedding operators of classical Sobolev spaces W_p^m as $m \to \infty$.

In the present article we obtain necessary and sufficient conditions for imbedding and compact imbedding, connected with the concrete structure of spaces $W^\infty\{a_n, p, r\}_{(G)}$ for the case of a rapidly decreasing sequence $\{a_n\}$, satisfying the condition

$$a_{n+1} \leqslant a_n^2 < 1, \quad n = 0, 1, \ldots \tag{1}$$

The results obtained below were announced in [4].

Moscow Energy Institute. Translated from Matematicheskie Zametki, Vol. 35, No. 4, pp. 505-516, April, 1984. Original article submitted May 26, 1981.

